Global Determinacy According to HANK*

David Murakami[†]

Ivan Shchapov[‡]

Yifan Zhang[§]

20 June 2025

Abstract

A linear rational expectations model that satisfies the Blanchard-Kahn conditions is deemed as locally determinate. If this model also possesses a unique minimum state variable (MSV) solution, we term it as "globally determinate". The canonical New Keynesian model subject to the effective lower bound (ELB) constraint does not generally possess a unique MSV solution unless monetary policy is passive; conversely violating local determinacy. This global indeterminacy problem stems from a strong feedback loop between expectations of endogenous variables and their current realisations at the ELB. This problem extends to a standard tractable heterogeneous agent New Keynesian (HANK) model. However, we show that global determinacy is restored under passive monetary policy and sufficiently limited asset market participation when "inverted aggregate demand logic" applies – further amplifying the "Catch-22 problem" in HANK models. Additionally, a standard HANK model with an active robust real rate rule fails to satisfy global determinacy conditions. But it is globally determinate with an inverted aggregate demand curve, much like the passive Taylor rule case.

Keywords: existence, uniqueness, effective lower bound, inverted aggregate demand

JEL Codes: C62, E4

^{*}We thank Guido Ascari for his guidance and supervision, and we thank Maria Eskelinen, Sophocles Mavroeidis, and seminar participants at the University of Oxford for helpful comments and feedback. Shchapov is grateful to *Chaire Finance Digitale* for funding his work at Institut Polytechnique de Paris.

[†]PhD candidate, University of Milan and University of Pavia; Visiting DPhil student, University of Oxford. david.murakami@unimi.it

[‡]PhD candidate, Institut Polytechnique de Paris, Centre for Research in Economics and Statistics (CREST), and Télécom Paris. Visiting PhD candidate at Johns Hopkins University ivan.shchapov@ensae.fr

[§]DPhil candidate, University of Oxford. yifan.zhang@economics.ox.ac.uk

1 Introduction

The representative agent New Keynesian (RANK) model with active monetary policy, subject to the effective lower bound (ELB), either features multiple minimum state variable (MSV) solutions with small shocks or solution non-existence with large shocks, as shown in Ascari and Mavroeidis (2022) and Holden (2023). Solution uniqueness can be achieved with passive monetary policy which, however, leads to indeterminate local dynamics – a violation of the Blanchard-Kahn (BK) conditions. More simply, a model that has a unique MSV solution that is locally determinate we refer to as satisfying "global determinacy". Our usage of the term global determinacy should not be confused with recent studies such as Ravn and Sterk (2020) and Acharya and Benhabib (2024), which refer to the existence of multiple steady states and the inability of the Taylor rule (TR) to pin down a unique rational expectations equilibrium when risk is sufficiently countercyclical.

Non-existence of an MSV solution is due to a strong feedback loop between current realisations of the endogenous variables and their expectations at the ELB. When monetary policy is constrained due to a large adverse shock, deflation materialises. Expectations of further deflation depress consumption and lead to more deflation and, as a consequence, higher real rates today, which results in a deflationary spiral. For the reference RANK model, one can ensure the existence of an MSV solution – without violating local determinacy – by, for example, introducing unconventional monetary policy (Ascari and Mavroeidis, 2022) (AM), deviating from rational expectations (Ascari, Mavroeidis, and McClung, 2023), or by introducing fiscal policy (Murakami, Shchapov, and Zhang, 2023).

This paper analyses whether household heterogeneity can provide a remedy for the non-uniqueness of equilibria in New Keynesian models. We find that MSV solution non-uniqueness persists in the tractable heterogeneous agent New Keynesian (HANK) model; this is whether the model is calibrated to match empirical evidence on cyclical income inequality (Patterson, 2023), or calibrated to resolve the "forwardguidance puzzle" (McKay, Nakamura, and Steinsson, 2016, 2017; Del Negro, Giannoni, and Patterson, 2023). Furthermore, we find the tractable HANK model only possesses a unique MSV solution that is locally determinate – and thus globally determinate – if it incorporates inverted aggregate demand logic (IADL) (Bilbiie, 2008); if the economy features a large enough proportion of hand-to-mouth (HtM) households and if the elasticity of their income with respect to aggregate income is large enough, the aggregate demand curve is inverted and, thus, an increase in real interest rates is associated with elevated consumption. Thus, when the economy is at the ELB and there is deflation, the HtM households increase their consumption which does not let the economy to descend into a deflationary spiral.

With a Taylor-type rule, IADL, however, implies that monetary policy's response to inflation has to be passive to ensure local determinacy (Bilbiie, 2008); the restrictions on the monetary policy feedback coefficient are cumbersome functions of structural parameters of the economy. We find that this could be overcome if the central bank resorts to a Robust Real Rate rule (RR), proposed by Holden (2024). Under a RR, the structure or curvature of aggregate demand has no bearing on the restrictions on the policy rule that ensure local determinacy and the standard Taylor principle applies. In other words, the combination of IADL and an RR allows the tractable HANK model to satisfy global determinacy with an active monetary policy rule.

This paper is related and contributes to three strands of literature: i) Multiple or non-existence of equilibria in the NK model subject to the ELB due to rational expectations (Benhabib, Schmitt-Grohé, and Uribe, 2001; Eggertsson and Woodford, 2003; Eggertsson, 2011; Mertens and Ravn, 2014; Boneva, Braun, and Waki, 2016; Armenter, 2017; Christiano, Eichenbaum, and Johannsen, 2018; Nakata, 2018; Nakata and Schmidt, 2019; Bilbiie, 2022; Ascari and Mavroeidis, 2022; Angeletos and Lian, 2023; Holden, 2023; Murakami, Shchapov, and Zhang, 2023); ii) HANK models and their tractable counterparts (Gornemann, Kuester, and Nakajima, 2016; McKay, Nakamura, and Steinsson, 2016, 2017; Kaplan, Moll, and Violante, 2018; Debortoli and Galí, 2017; Kaplan, Moll, and Violante, 2018; Ravn and Sterk, 2020; Bilbiie, 2020, 2024); and iii) weakening the "rationality" of households to augment the household Euler equation (Angeletos and Lian, 2018; Gabaix, 2020; Ascari, Mavroeidis, and McClung, 2023; Del Negro, Giannoni, and Patterson, 2023).

The paper proceeds as follows. Section 2 outlines the model environment, Section 2.1 outlines the MSV solution non-uniqueness problem inherent in HANK models with standard aggregate demand logic and illustrates that this problem is not present with IADL. Section 2.2 demonstrates the implications of a robust real rate rule for global determinacy. Section 3 concludes the paper.

2 Reference HANK Model

Consider the following tractable HANK model as in Bilbiie (2020, 2024) and Debortoli and Galí (2024):

$$x_t = \delta \mathbb{E}_t x_{t+1} - \widehat{\sigma}(i_t - \mathbb{E}_t \pi_{t+1}) + \varepsilon_t, \tag{1}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t, \tag{2}$$

where variables are expressed as log-deviations from their deterministic steady state values: x_t is the output gap, π_t is inflation, i_t is the nominal interest rate, ε_t is a demand shock, β is the household discount factor, (1) is the dynamic IS equation (DISE) or aggregate Euler equation, and (2) is the New Keynesian Phillips Curve (NKPC). To close the model we first assume a standard Taylor-type rule,

$$i_t = \max\left\{-\mu, \phi \pi_t\right\},\tag{3}$$

with $\mu = \pi^* + \rho$, π^* is the net inflation target, and ρ is the discount rate (such that $\frac{1}{\beta} = 1 + \rho$). The expressions for δ and $\hat{\sigma}$ in the DISE are given as:

$$\delta \equiv 1 + \frac{(\chi - 1)(1 - s)}{1 - \lambda \chi}, \quad \widehat{\sigma} \equiv \frac{1 - \lambda}{\sigma (1 - \lambda \chi)}, \tag{4}$$

where χ denotes the elasticity of a "hand-to-mouth/restricted" household's consumption and income to aggregate income, *s* is a switching probability of an "unrestricted" household (an agent on their consumption Euler equation) remaining as an unrestricted household, and λ is the mass of restricted households. Note that if $\hat{\sigma} = \frac{1}{\sigma}$ (the intertemporal elasticity of substitution), and $\delta = 1$, then the DISE is equivalent to its textbook representative-agent form as in Woodford (2003) and Galí (2015).

2.1 Global Determinacy with a Taylor-type rule: Catch-22 (Again)

We stipulate that global determinacy for a linear forward-looking rational expectations model requires satisfying both local determinacy (Blanchard-Kahn conditions) and for the model to possess a unique MSV solution. To verify the latter, we follow the method of Ascari and Mavroeidis (2022) and illustrate the intuition following Murakami, Shchapov, and Zhang (2023). As in Eggertsson and Woodford (2003), Nakata and Schmidt (2019), Christiano, Eichenbaum, and Johannsen (2018), and Ascari and

Mavroeidis (2022), we begin by assuming that the demand or preference shock follows a two-state Markov process with states $\varepsilon_t = \{\varepsilon^T, 0\}$ and transition matrix

$$K = \begin{bmatrix} p & 1-p\\ 1-q & q \end{bmatrix},\tag{5}$$

where p is the probability of remaining in the first state and q is the probability of remaining in the second state. With this, we summarise global determinacy for the tractable HANK model with a Taylor-type rule in Proposition 1:

Proposition 1 The tractable HANK model with a Taylor-type rule, as in Equations (1)-(3), satisfies global determinacy if it concurrently satisfies local determinacy (Blanchard-Kahn conditions) and if it possesses a unique minimum state variable solution. The latter of which holds if

1. either
$$\widehat{\sigma} < 0$$
, and $\phi - \vartheta < 0$, and $\delta < 1 - \frac{\kappa(\phi-1)}{1-\beta} |\widehat{\sigma}|$;

2. or $\widehat{\sigma} > 0$, and $\vartheta < 0$, and $\delta < 1 - \frac{\kappa \widehat{\sigma}}{1 - \beta}$;

and where

$$\vartheta \equiv p + q - 1 - \frac{\left[(1 - \delta) + \delta \left(2 - p - q\right)\right] \left(1 - \beta q - \beta p + \beta\right)}{\kappa \widehat{\sigma}}.$$

Proof: Appendix A.1.

To explain the intuition, we assume that q = 1 for analytical tractability, and that the model's absorbing state is the positive interest rate (PIR) steady state, $\{x_t, \pi_t, i_t\} = \{0, 0, 0\}$. As such, with rational expectations we can write $\mathbb{E}_t \pi_{t+1} = p \pi_t$ to express the aggregate Euler equation (1) as:

$$x_t = \delta \mathbb{E}_t x_{t+1} - \widehat{\sigma} \left(\max \left\{ -\mu, \phi \pi_t \right\} - p \pi_t \right) + \varepsilon_t$$

Substituting the NKPC (2) into the above equation, after writing it as $\pi_t(x_t)$, gives:

$$x_t = \left(\delta + \frac{\widehat{\sigma}\kappa}{1 - p\beta}\right) \mathbb{E}_t x_{t+1} - \widehat{\sigma} \max\left\{-\mu, \frac{\phi\kappa}{1 - p\beta}x_t\right\} + \varepsilon_t.$$

We thus get two forward-looking difference equations which correspond to the con-

strained and unconstrained regimes,

$$x_t = \Lambda_0 \mathbb{E}_t x_{t+1} + \widehat{\sigma} \mu + \varepsilon_t, \tag{6}$$

$$x_t = \Lambda_1 \mathbb{E}_t x_{t+1} + \Upsilon \varepsilon_t, \tag{7}$$

respectively, and where

$$\Lambda_0 \equiv \delta + \frac{\widehat{\sigma}\kappa}{1 - p\beta},\tag{8}$$

$$\Lambda_1 \equiv \frac{\delta(1-p\beta) + \widehat{\sigma}\kappa}{1-p\beta + \widehat{\sigma}\kappa\phi},\tag{9}$$

with $\Upsilon \equiv \frac{1-p\beta}{1-p\beta+\hat{\sigma}\kappa\phi}$. First, consider the unconstrained forward-looking difference equation (7). It possesses a fundamental solution if $|p\Lambda_1| < 1$. In conventional analysis of DSGE models, this condition corresponds to the satisfaction of the BK condition.¹ In a RANK model, the BK condition is satisfied if the monetary policy rule satisfies the Taylor principle, i.e., $\phi > 1$. For the baseline tractable HANK model, with $\lambda \chi < 1 \implies \hat{\sigma} > 0$, from (7) the BK condition is satisfied if the following HANK Taylor principle holds (Bilbiie, 2024):

$$\phi > 1 + \frac{(1 - p\beta)(\delta - 1)}{\widehat{\sigma}\kappa}.$$

Note that for the special case of q = 1, that ϑ in Proposition 1 represents the lower bound for the HANK Taylor principle.

It follows then that we require $|p\Lambda_0| < 1$ in (6) in order for a fundamental solution – the unique MSV solution – to exist for the constrained forward-looking difference equation.² The intuition for this is as follows: If $|p\Lambda_0| > 1$ then (6) implies that a nega-

$$x_t = \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_0^s \left(\widehat{\sigma} \mu + \varepsilon_{t+s} \right),$$

where the fundamental solution is:

$$x_t = \frac{\widehat{\sigma}}{1 - \Lambda_0} \mu + \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_0^s \varepsilon_{t+s}.$$

^{1.} Recall that the root of the characteristic polynomial is $(|\delta \Lambda_1|)^{-1}$. Since this is a single equation with a forward-looking variable, in order to satisfy the BK condition, we require the root to be outside of unit circle.

^{2.} Since this corresponds to the case of continuous shock support:

| | Implied parameters | | | | | | |
|---|--------------------|------|------|------|--------------------|-------------|-------------|
| | χ | λ | S | δ | $\widehat{\sigma}$ | Λ_0 | Λ_1 |
| HANK-C | | | | | | | |
| Gornemann, Kuester, and Nakajima (2016) | 1.76 | 0.30 | 0.96 | 1.06 | 0.74 | 14.18 | 0.52 |
| Debortoli and Galí (2017) | 2.38 | 0.21 | 0.96 | 1.11 | 0.79 | 15.08 | 0.52 |
| Kaplan, Moll, and Violante (2018) | 1.42 | 0.41 | 0.96 | 1.04 | 0.71 | 13.53 | 0.52 |
| HANK-P | | | | | | | |
| McKay, Nakamura, and Steinsson (2016, 2017) | 0.30 | 0.21 | 0.96 | 0.97 | 0.42 | 8.43 | 0.53 |

Table 1: Calibration of HANK Models

Note: Parameter values sourced from Bilbiie (2020) and authors' calculations. Additional parameter calibration values are $\phi = 2$, $\beta = 0.99$, $\sigma = 2$, $\kappa = 0.1769$, and p = 1 for Λ_0 and Λ_1 calculations.

tive output gap will lead to a more negative expected output gap $\mathbb{E}_t x_{t+1}$ for a negative demand shock $\varepsilon^T < 0$. This is the negative feedback loop we showed in Murakami, Shchapov, and Zhang (2023), whereby the loop leads to no fixed point where expectations settle into an equilibrium. Another way to see this is to use $\mathbb{E}_t x_{t+1} = px_t$ to write (6) as:

$$x_t = \frac{1}{1 - p\Lambda_0} \left(\widehat{\sigma} \mu + \varepsilon_t \right),$$

and one can see that if $|p\Lambda_0| > 1$, a large negative realisation of ε^T leads to x_t being positive, which is inconsistent with an ELB equilibrium. For the RANK model, and as discussed in Ascari and Mavroeidis (2022), for a fundamental solution to exist it is required that either: i) p be sufficiently small so that $|p\Lambda_0| < 1$, or ii) that ε^T be sufficiently small, i.e., have sufficient mass around zero as discussed in Holden (2023). Here in the HANK model, the requirement $|p\Lambda_0| < 1$ is more difficult to assess because of δ and $\hat{\sigma}$ – specifically the relationship between the two due to $\lambda \chi$.

We argue that global determinacy, in a HANK model subject to the ELB, requires that there exist a fundamental solution in both the constrained and unconstrained forward-looking difference equations, (6) and (7). In other words, global determinacy requires $\{|\Lambda_0|, |\Lambda_1|\} < 1$.

We test if the tractable HANK model satisfies global determinacy with two classes of standard calibrations, which we refer to as "HANK-C" and "HANK-P", as shown in Table 1. These specifications coincide to the "Catch-22" for the reference HANK model adopted in the literature: the trade-off between countercyclical income inequality consistent with empirical evidence in Patterson (2023) (HANK-C) and resolution

of the forward guidance puzzle with procyclical income inequality as in McKay, Nakamura, and Steinsson (2016, 2017) (MNS) (HANK-P).³ Under these calibrated values, we compute the coefficients of the forward-looking difference equations (6) and (7), setting p = 1 and display the results in Table 1.

Despite satisfying local determinacy requirements ($|\Lambda_1| < 1$), both HANK-C and HANK-P fail to satisfy global determinacy requirements, because the coefficient in the constrained regime forward-looking difference equation (6) is greater than unity, i.e., $|\Lambda_0| > 1$. The reason is a dominant income effect channel, since it exerts upward pressure on inflation and output following a negative demand shock $\varepsilon^T < 0$: an increase in $\mathbb{E}_t x_{t+1}$ leads to a greater increase in x_t . This is a potential non-existence of an MSV solution at the ELB. For the case of the RANK model, this problem can be addressed by, for example, introducing unconventional monetary policy (Ascari and Mavroeidis, 2022), deviating from rational expectations (Ascari, Mavroeidis, and McClung, 2023), or by introducing simple fiscal policy (Murakami, Shchapov, and Zhang, 2023) to reduce the strength of the income-effect feedback loop between $\mathbb{E}_t x_{t+1}$ and x_t by lowering the value of Λ_0 .

But in the tractable HANK framework, the presence of restricted households, as well as the switching probability for an unrestricted households to become a restricted household, provides a potential remedy for the non-existence of an MSV solution. Combined with satisfaction of the BK conditions, this will allow the HANK model to satisfy global determinacy requirements. For instance, by keeping all parameter values except σ fixed as in Table 1, only the HANK-P calibration of MNS can satisfy global determinacy.⁴ However, this requires an extremely high level of risk aversion $\sigma \ge 500$,⁵ an implausible value in the empirical macroeconomics literature.

One way to proceed is to consider the case in which $\lambda \chi > 1$. Recall that since λ is the share of restricted households and if $\lambda \in (0, 1)$, we are considering the case of countercyclical income inequality, $\chi > 1/\lambda$.⁶ Under this condition we have $\delta < 1$ and $\hat{\sigma} < 0$, and are subject to the inverted Keynesian cross or "inverted aggregate demand

5. The condition $|\Lambda_0| < 1$ is given by

$$\sigma > \frac{(1-\lambda)\kappa}{(1-\lambda\chi)(1-\delta)(1-p\beta)},$$

in the case of $\delta < 1$ and p = 1.

6. Note that the class of HANK-C models are ruled out.

^{3.} See Bilbiie (2024) for details.

^{4.} Since in the limit, as $\hat{\sigma} \to 0$, $\Lambda_0 \to \delta$ and $\Lambda_1 \to \delta$, HANK-C fails to satisfy BK conditions and does not possess a unique MSV solution.

logic" (IADL) outlined in Bilbiie (2008, 2024). With IADL, increases in the real interest rate correspond to increases in consumption, thereby leading to an increase in x_t .

Global determinacy with IADL. Since we are not interested in solutions that cause oscillations, we require $\Lambda_0^{-1} > 1$, which we can write as:

$$\delta < 1 - \frac{\widehat{\sigma}\kappa}{1 - p\beta'}$$

which always holds true if $\chi \lambda > 1 \implies \delta < 1$.

For $\Lambda_1^{-1} > 1$ from (9), we it needs to hold that

$$\phi < 1 + \frac{(1 - p\beta)(\delta - 1)}{\widehat{\sigma}\kappa},\tag{10}$$

which implies that the monetary policy feedback to inflation has to be passive.⁷

To summarise these results: a HANK-C model with an active TR generally fails to satisfy the requirements to possess a unique MSV solution, much like the RANK model with an active TR. Setting a passive TR in the HANK-C model allows it to possess a unique MSV solution, but at the cost of violating local determinacy (BK) conditions. This case corresponds to the red region below the dashed $\chi\lambda = 1$ line in Figure 1, and which was thoroughly discussed by Ascari and Mavroeidis (2022) for the RANK model. It is thus not a candidate for satisfying what we call global determinacy.

However, a HANK-P model that is subject to IADL and a passive TR can satisfy both the requirements of possessing a unique MSV solution and local determinacy. This corresponds to the blue region above the dashed $\chi \lambda = 1$ line in Figure 1. We thus claim that HANK-P models are candidates to satisfy global determinacy. But, the calibration of MNS, despite being a HANK-P class model, does not satisfy global determinacy due to not having IADL as $\lambda \chi < 1$.

Hence, the main contention of this paper is that requirements of global determinacy further amplify the Catch-22 of [tractable] HANK models described by Bilbiie (2020). HANK-C aggravates the forward guidance puzzle and is globally indeterminate despite featuring consumption inequality cyclicality in line with empirical evidence; HANK-P can resolve this puzzle, along with satisfying global determinacy when properly calibrated, but the calibrations are not empirically plausible.

^{7.} In principle, the denominator of Λ_0^{-1} is ambiguous; it is, however, non-negative under any reasonable calibration of κ and s.



Figure 1: Global Determinacy Region for the HANK Model with a Taylor Rule

Note: Simulations run with $\beta = 0.99$, $\sigma = 2$, $\kappa = 0.1769$, $\phi = 0.5$, s = 0.96 and p = 1. The blue region above the dashed line corresponds to inverted aggregate demand logic case ($\lambda \chi = 1$), and where the model possesses global determinacy. The red region below the dashed line results in the model being locally indeterminate (BK condition violation).

Illustrating the HANK global indeterminacy problem. To illustrate the global determinacy challenge for the TR case, assume that the economy is initially in a transitory state, $\{x_t, \pi_t\} = \{x^T, \pi^T\}$, and remains there with probability p and transitions to the absorbing PIR steady state with complementary probability 1 - p. This transitory state is driven by the preference shock described by (5). Hence, the HANK model can be

^{8.} In other words, off of the positive interest rate steady state.

Figure 2: Non-Uniqueness or Non-Existence of an MSV Solution in the HANK Model (with Taylor Rule) in Transitory State



Note: Figure shows AS (blue) and piecewise-linear AD (red) curves. Left panel shows the the problem of non-existence and multiplicity of equilibria with varying shock sizes under HANK-C calibration. Right panel shows equilibrium uniqueness under the inverted aggregate demand.

expressed using aggregate demand (AD) and aggregate supply (AS) schedules:

$$\pi^{T} = \begin{cases} \frac{1-p\delta}{\widehat{\sigma}(p-\phi)} x^{T} - \frac{\varepsilon^{T}}{\widehat{\sigma}(p-\phi)} & AD^{TR}, \\ \frac{1-p\delta}{p\widehat{\sigma}} x^{T} - \frac{\mu}{p} - \frac{\varepsilon^{T}}{p\widehat{\sigma}} & AD^{ELB}, \end{cases}$$
(11)

$$\pi^T = \frac{\kappa}{1 - p\beta} x^T \quad AS.$$
(12)

Figure 2 plots the *AD* and *AS* schedules for the tractable HANK model. Subfigure 2a shows the issue of MSV solution multiplicity (shown for AD^1 -AS) or non-existence (shown for AD^2 -AS) under HANK-C. For small realisations of the shock, *AD* intersects *AS* twice – a case of multiple MSV solutions or incompleteness in the terminology of Gourieroux, Laffont, and Monfort (1980) and Ascari and Mavroeidis (2022). Conversely, for sufficiently large negative realisations of ε^T , no MSV solution exists since neither $AD^{TR,2}$, $AD^{ELB,2}$ intersect *AS*.⁹ If monetary policy is passive, such that the slope of AD^{TR} is flatter than *AS*, then there exists at least one MSV solution. But, to reiterate, this violates local determinacy; corresponding to the red region below the $\lambda \chi = 1$ dashed line in Figure 1.

^{9.} Note that in Figure 2 we set p to be sufficiently small. If p is high enough, then AD^{TR} becomes upward sloping. However, qualitatively, the MSV solution problem remains.

Conversely, Subfigure 2b plots *AD* and *AS* for the HANK-P model under IADL, showing the existence of a unique MSV solution. No matter the size of the shock there always exists a unique MSV solution, and since the *AD* is inverted there is no BK condition violation for a passive monetary policy rule in the unconstrained regime. Thus, Subfigure 2b corresponds to the blue region above the $\lambda \chi = 1$ dashed line in Figure 1.

2.2 HANK with a Robust Real Rate Rule

We have previously shown that global determinacy is achieved in a tractable HANK model under IADL. This, however, implies cumbersome restrictions on the monetary policy feedback coefficients; monetary policy has to respond passively to inflation. Below we show that this problem can be overcome by a central bank adopting a Robust Real Rate rule (RR) as in Holden (2024).

The RR takes the following form:

$$i_t = \max\{-\mu, r_t + \phi \pi_t\}.$$
(13)

Combined with the Fisher equation absent the ELB, this yields:

$$r_t + \mathbb{E}_t \pi_{t+1} = r_t + \phi \pi_t \implies \mathbb{E}_t \pi_{t+1} = \phi \pi_t,$$

which implies a unique non-explosive solution with $\pi_t = 0$ if $\phi > 1$, which represents the key property of the rule.¹⁰ Here, we emphasise the robustness of the rule in the sense that it implies determinacy with active monetary policy regardless of the form of the aggregate demand condition. With the RR, aggregate demand or the aggregate Euler equation is irrelevant for the determination of inflation or the output gap; and hence it also has no bearing on satisfaction of the BK conditions. Thus, with the RR, we need only concern ourselves with having a unique MSV solution, which leads us to Proposition 2:

Proposition 2 The tractable HANK model with a Robust Real Rate rule, as in Equations (1), (2), and (13), satisfies global determinacy if it concurrently satisfies local determinacy (Blanchard-Kahn conditions) and if it possesses a unique minimum state variable solution. The latter of which holds if $\hat{\sigma} < 0$ and $0 < \delta < 1$.

^{10.} This of course implies that the real interest rate adjusts in response to the preference shocks, $r_t = \hat{\sigma}^{-1} \varepsilon_t$. For further discussion on the robustness properties of this rule, recommend reading Holden (2024).

Figure 3: Global Determinacy Region for the HANK Model with a Robust Real Rate Rule



Note: Simulations run with $\beta = 0.99$, $\sigma = 2$, $\kappa = 0.1769$, $\phi = 2$, s = 0.96 and p = 1. The blue region above the dashed line corresponds to inverted aggregate demand logic case ($\lambda \chi = 1$), and where the model possesses global determinacy. The yellow region below the dashed line results in the model not possessing a unique minimum state variable solution.

Proof: Appendix A.2.

An [active] RR ensures that the MSV solution to the model is unique and locally determinate as long as IADL holds. As mentioned above, aggregate demand is unimportant for the dynamics of the model when the interest rate is unconstrained; inflation and output gaps remain closed as long as the central bank is able to accommodate all demand shock by adjusting the real rate.

When the demand shocks are large enough to bring the economy to the ELB, aggregate demand becomes important in pinning down equilibrium outcomes. Normal aggregate demand logic, $\hat{\sigma} > 0$, implies that deflation and real rate increases at the Figure 4: Non-Uniqueness or Non-Existence of an MSV Solution in the HANK Model (with Robust Real Rate Rule) in Transitory State



Note: Figure shows AS (blue) and piecewise-linear AD (red) curves with Robust Real Rate rule. Left panel shows the problem of non-existence and multiplicity of equilibria with varying shock sizes with normal aggregate demand. Right panel shows equilibrium uniqueness under the inverted aggregate demand. $\underline{x} = \mu \hat{\sigma}/(1 - p\delta)$

ELB lead to a decline in output, which leads to further deflation forming a deflationary spiral. This is the source of equilibrium non-existence at the ELB in NK models when shocks are large enough.

Under IADL, however, increases in real rates do not lead to a decline in real activity. On the contrary, the rising real rates due to deflation at the ELB imply higher economic activity and thus higher inflation. Thus, the agents do not expect deflation in equilibrium at the ELB and deflationary spirals never materialise. Similar to the TR case, we plot the global determinacy region for the RR case in Figure 3. The blue region above the dashed $\lambda \chi = 1$ line corresponds to the IADL case and thus satisfaction of global determinacy requirements, where as the yellow region below the dashed line is where only local determinacy holds but with no or multiple MSV solutions.

To illustrate non-existence or multiplicity of MSV solutions in the HANK model with RR, we again write the model in terms of *AD* and *AS* schedules:

$$\pi^{T} = \begin{cases} 0 & AD^{RR}, \\ \frac{(1-p\delta)}{p\widehat{\sigma}} x^{T} - \frac{\mu}{p} - \frac{\varepsilon^{T}}{p\widehat{\sigma}} & AD^{ELB}, \end{cases}$$
(14)

and where *AS* is given as before in Equation (12). We then plot the relations in Figure 4. Figure 4a shows the aggregate demand and supply schedules in the transitory state under normal aggregate demand logic. For high values of *p*, *AD*^{*ELB*} is flatter than *AS* which implies that there exists a ZIR and a PIR equilibria.¹¹ Figure 4a thus corresponds to the yellow region below $\lambda \chi = 1$ in Figure 3.

Subfigure 4b shows the IADL case where aggregate demand is downward sloping when monetary policy is constrained. The intersection of AD^{ELB} lies to the right from the cut-off <u>x</u> and is thus inconsistent with the ELB constraint.¹² Since monetary policy given by RR is active – satisfying local determinacy – the model also possesses a unique MSV solution and is thus globally determinate. Hence, Subfigure 4b corresponds to the blue region above $\lambda \chi = 1$ in Figure 3.

3 Conclusion

We show that the tractable HANK model, when subject to the ELB, fails to yield a unique MSV solution if calibrated to match empirical evidence on cyclical income inequality. We refer to this as a failure of global determinacy, defined as satisfying both local determinacy and uniqueness of the MSV solution. However, global determinacy can be restored under empirically implausible values of the intertemporal elasticity of substitution, especially when the model is also calibrated to resolve the forward guidance puzzle — thus intensifying the Catch-22 highlighted by Bilbiie (2020) and others.

We also show that global determinacy can be achieved under a passive Taylor rule if the economy features a high share of HtM households with strongly procyclical incomes (an inverted aggregate demand curve). This reverses the standard relationship between real interest rates and consumption, ruling out self-fulfilling deflationary spirals at the ELB and introducing a novel Catch-22 not previously discussed in the literature.

Finally, we find that global determinacy with an inverted demand curve can coexist with active monetary policy if the central bank adopts a Robust Real Rate rule, as proposed by Holden (2024).

^{11.} To find this value, solve the following inequality for $p: \frac{\kappa}{1-\beta p} > \frac{1-p\delta}{p\sigma} \implies \beta \delta p^2 - p(\beta + \delta + \kappa \hat{\sigma}) + 1 < 0$. The inequality is quadratic in p and implies that if p is large enough, AD^{ELB} is upward-sloping and flatter than AS.

^{12.} The AD^{ELB} and AS curves intersect at $x^* = -\frac{\mu}{\Psi}$, where $\Psi \equiv \frac{(p\delta-1)(1-p\beta)+p\widehat{\sigma}\kappa}{\widehat{\sigma}(1-p\beta)}$. Under IADL, it always holds that $x^* > x$.

References

- Acharya, Sushant, and Jess Benhabib. 2024. Global Indeterminacy in HANK Economies. Working Paper, Working Paper Series 32462. National Bureau of Economic Research. https://doi.org/10.3386/w32462. http://www.nber.org/papers/w32 462.
- Angeletos, George-Marios, and Chen Lian. 2018. "Forward Guidance without Common Knowledge." *American Economic Review* 108 (9): 2477–2512.

2023. "Determinacy without the Taylor Principle." *Journal of Political Economy* 131 (8): 2125–2164. https://doi.org/10.1086/723634.

- Armenter, Roc. 2017. "The Perils of Nominal Targets." *The Review of Economic Studies* 85 (1): 50–86. https://doi.org/10.1093/restud/rdx001.
- Ascari, Guido, and Sophocles Mavroeidis. 2022. "The Unbearable Lightness of Equilibria in a Low Interest Rate Environment." *Journal of Monetary Economics* 27:1–27.
- Ascari, Guido, Sophocles Mavroeidis, and Nigel McClung. 2023. "Coherence without Rationality at the Zero Lower Bound." *Journal of Economic Theory* 214:105745. http s://doi.org/https://doi.org/10.1016/j.jet.2023.105745.
- **Benhabib**, Jess, Stephanie Schmitt-Grohé, and Martín Uribe. 2001. "Monetary Policy and Multiple Equilibria." *American Economic Review* 91 (1): 167–186.
- Bilbiie, Florin O. 2008. "Limited Asset Markets Participation, Monetary Policy and (inverted) Aggregate Demand Logic." *Journal of Economic Theory* 140 (1): 162–196. https://doi.org/https://doi.org/10.1016/j.jet.2007.07.008.

——. 2020. "The New Keynesian Cross." Journal of Monetary Economics 114:90–108.

 2022. "Neo-Fisherian Policies and Liquidity Traps." American Economic Journal: Macroeconomics 14, no. 4 (October): 378–403. https://doi.org/10.1257/mac.202001
 19. https://doi.org/10.1257/mac.20200119.

——. 2024. "Monetary Policy and Heterogeneity: An Analytical Framework." Review of Economic Studies (forthcoming).

- Boneva, Lena Mareen, R. Anton Braun, and Yuichiro Waki. 2016. "Some Unpleasant Properties of Loglinearized Solutions when the Nominal Rate is Zero." *Journal of Monetary Economics* 84:216–232.
- Christiano, Lawrence, Martin Eichenbaum, and Benjamin Johannsen. 2018. "Does the New Keynesian Model Have a Uniqueness Problem?" *NBER Working Paper*, https://doi.org/10.3386/w24612. https://doi.org/10.3386/w24612.
- **Debortoli, Davide, and Jordi Galí.** 2017. *Monetary Policy with Heterogeneous Agents: Insights from TANK Models.* Technical report. Department of Economics and Business, Universitat Pompeu Fabra.

———. 2024. "Idiosyncratic Income Risk and Aggregate Fluctuations." *American Economic Journal: Macroeconomics (forthcoming)*.

- **Del Negro, Marco, Marc P. Giannoni, and Christina Patterson.** 2023. "The Forward Guidance Puzzle." *Journal of Political Economy Macroeconomics* 1 (1): 43–79. https://doi.org/10.1086/724214.
- **Eggertsson, Gauti B.** 2011. "What Fiscal Policy is Effective at Zero Interest Rates?" *NBER Macroeconomics Annual* 25:59–112.
- **Eggertsson, Gauti B., and Michael Woodford.** 2003. "The Zero Bound on Interest Rates and Optimal Monetary Policy." *Brookings Papers on Economic Activity* 2003 (1): 139– 211.
- Gabaix, Xavier. 2020. "A Behavioral New Keynesian Model." *American Economic Review* 110 (8): 2271–2327. https://doi.org/10.1257/aer.20162005.
- **Galí, Jordi.** 2015. *Monetary Policy, Inflation, and the Business Cycle*. 2nd Edition. Princeton University Press.
- Gornemann, Nils, Keith Kuester, and Makoto Nakajima. 2016. "Doves for the Rich, Hawks for the Poor? Distributional Consequences of Systematic Monetary Policy." *Federal Reserve Bank of Minneapolis Institute Working Paper Series*, no. 50, https://do i.org/https://doi.org/10.21034/iwp.50.
- **Gourieroux, Christian, Jean-Jacques Laffont, and Alain Monfort.** 1980. "Coherency Conditions in Simultaneous Linear Equation Models with Endogenous Switching Regimes." *Econometrica*, 675–695.

- Holden, Tom D. 2023. "Existence and Uniqueness of Solutions to Dynamic Models with Occasionally Binding Constraints." *The Review of Economics and Statistics* 105 (6): 1481–1499. https://doi.org/10.1162/rest_a_01122.
- Kaplan, Greg, Benjamin Moll, and Giovanni L. Violante. 2018. "Monetary Policy According to HANK." *American Economic Review* 108 (3): 697–743.
- McKay, Alisdair, Emi Nakamura, and Jón Steinsson. 2016. "The Power of Forward Guidance Revisited." *American Economic Review* 106 (10): 3133–3158.

——. 2017. "The Discounted Euler Equation: A Note." *Economica* 84 (336): 820–831. https://doi.org/https://doi.org/10.1111/ecca.12226.

- Mertens, Karl R. S. M., and Morten O. Ravn. 2014. "Fiscal Policy in an Expectations-Driven Liquidity Trap." *The Review of Economic Studies* 81 (4): 1637–1667.
- Murakami, David, Ivan Shchapov, and Yifan Zhang. 2023. Restoring Existence and Uniqueness at the Effective Lower Bound with Simple Fiscal Policy. Technical report. working paper. https://doi.org/https://dx.doi.org/10.2139/ssrn.4326398.
- **Nakata, Taisuke.** 2018. "Reputation and Liquidity Traps." *Review of Economic Dynamics* 28 (252-268).
- Nakata, Taisuke, and Sebastian Schmidt. 2019. "Conservatism and Liquidity Traps." Journal of Monetary Economics 104:37–47.
- Patterson, Christina. 2023. "The Matching Multiplier and the Amplification of Recessions." American Economic Review 113 (4): 982–1012. https://doi.org/10.1257/aer .20210254.
- Ravn, Morten O, and Vincent Sterk. 2020. "Macroeconomic Fluctuations with HANK & SAM: an Analytical Approach." *Journal of the European Economic Association* 19 (2): 1162–1202. ISSN: 1542-4766. https://doi.org/10.1093/jeea/jvaa028.
- Woodford, Michael. 2003. Interest and Prices. Princeton University Press.

A Existence and Uniqueness of an MSV Solution

A.1 Taylor-type rule

We now outline a sketch of verifying existence and uniqueness of a MSV solution – which we refer to as global determinacy – for the reference HANK model with a Taylor-type rule. Following (3), we refer to $i_t = -\mu$ as the case where the ELB binds ($s_t = 0$), and $i_t > -\mu_t$ as the unconstrained case ($s_t = 1$). The model can then be written as a system of difference equations:

$$A_{s_t}Y_t + B_{s_t}Y_{t+1|t} + C_{s_t}X_t + D_{s_t}X_{t+1|t} = \mathbf{0},$$
(15)

where Y and X denote vectors of endogenous and exogenous variables, respectively. We thus have:

$$\underbrace{\begin{bmatrix} 1 & \widehat{\sigma}\phi \\ -\kappa & 1 \end{bmatrix}}_{A_1} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} -\delta & -\widehat{\sigma} \\ 0 & -\beta \end{bmatrix}}_{B_1} \begin{bmatrix} \mathbb{E}_t x_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}}_{C_1} \begin{bmatrix} \varepsilon_t \\ \mu \end{bmatrix} = \mathbf{0}, \quad \text{for } i_t > -\mu.$$
(16)

and

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix}}_{A_0} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} -\delta & -\widehat{\sigma} \\ 0 & -\beta \end{bmatrix}}_{B_0} \begin{bmatrix} \mathbb{E}_t x_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & -\widehat{\sigma} \\ 0 & 0 \end{bmatrix}}_{C_0} \begin{bmatrix} \varepsilon_t \\ \mu \end{bmatrix} = \mathbf{0}, \quad \text{for } i_t = -\mu, \quad (17)$$

Assume that ε_t follows a *k*-state Markov process with k = 2, where *K* is a *k*-state transition matrix,¹³ and where I_k is an identity matrix with columns e_i . Following Theorem 1 from Gourieroux, Laffont, and Monfort (1980) (GLM) and AM, existence and uniqueness of an MSV solution requires that the signs of det \mathcal{A}_J , $J \subseteq \{1, ..., k\}$ are identical, where

$$\boldsymbol{K} = \begin{bmatrix} p & 1-p\\ 1-q & q \end{bmatrix}.$$

^{13.} *K* has switching probabilities *p* and *q*:

$$\mathcal{A}_{J_1} = \mathbf{I}_2 \otimes A_1 + \mathbf{K} \otimes \mathbf{B}_1 \qquad J_1 = \{1, 2\} \quad (\text{PIR,PIR}),$$

$$\mathcal{A}_{J_2} = \mathbf{e}_1 \mathbf{e}_1^\top \otimes A_0 + \mathbf{e}_1 \mathbf{e}_1^\top \mathbf{K} \otimes \mathbf{B}_0 + \mathbf{e}_2 \mathbf{e}_2^\top \otimes A_1 + \mathbf{e}_2 \mathbf{e}_2^\top \mathbf{K} \otimes \mathbf{B}_1 \qquad J_2 = \{2\} \quad (\text{ZIR,PIR}),$$

$$\mathcal{A}_{J_3} = \mathbf{e}_1 \mathbf{e}_1^\top \otimes A_1 + \mathbf{e}_1 \mathbf{e}_1^\top \mathbf{K} \otimes \mathbf{B}_1 + \mathbf{e}_2 \mathbf{e}_2^\top \otimes A_0 + \mathbf{e}_2 \mathbf{e}_2^\top \mathbf{K} \otimes \mathbf{B}_0 \qquad J_3 = \{1\} \quad (\text{PIR,ZIR}),$$

$$\mathcal{A}_{J_4} = \mathbf{I}_2 \otimes A_0 + \mathbf{K} \otimes \mathbf{B}_0, \qquad J_4 = \emptyset \quad (\text{ZIR,ZIR}),$$

$$(18)$$

where *J* denotes whether the ELB constraint is slack or not for the exogenous states of ε_t . Specifically, $J_1 = \{1, 2\}$ implies that the constraints are slack in both states ($s_i = 1$ for $i = \{1, 2\}$). $J_2 = \{2\}$ implies that the constraint is slack in state i = 2 but binding in state i = 1 ($s_2 = 1$ and $s_1 = 0$). $J_3 = \{1\}$ implies that the constraint is slack in state i = 1 but binding in state i = 2 ($s_1 = 1$ and $s_2 = 0$). $J_4 = \emptyset$ implies that the constraint is binding in both states (s = 0 for $i = \{1, 2\}$).¹⁴

To analytically derive (18), start by writing the Euler equation using the two-state Markov chain process for ε_t and *K*:

 $x = \delta K x - \widehat{\sigma} \left(R - K \pi \right) + \varepsilon,$

and then rearrange:

$$(I - \delta K) x = -\widehat{\sigma} (R - K\pi) + \varepsilon.$$
⁽¹⁹⁾

While the NKPC can be written as:

 $\pi = \beta K \pi + \kappa x,$

and multiplying it by $(I - \delta K)$ yields:

$$(I - \delta K)\pi = (I - \delta K)\beta K\pi + \kappa (I - \delta K)x.$$
⁽²⁰⁾

19

^{14.} Note that since we only have one inequality constraint in the model, we need not use the Kronecker product for computing the matrices in (18).

Combining (19) and (20) together gives:

$$(I - \delta K) \pi = (I - \delta K) \beta K \pi + \kappa \left[-\widehat{\sigma} (R - K \pi) + \varepsilon \right]$$
$$\underbrace{\left[(I - \delta K) - \beta (I - \delta K) K - \kappa \widehat{\sigma} K \right]}_{\equiv Q} \pi = \kappa \left(-\widehat{\sigma} R + \varepsilon \right)$$

The Taylor rule is

$$\boldsymbol{R}=\max\left\{-\mu\boldsymbol{\iota}_{2},\phi\boldsymbol{\pi}\right\},\,$$

and substituting into the equation above gives:

$$Q\pi = -\kappa \widehat{\sigma} \max\left\{-\mu \iota_2, \phi \pi\right\} + \kappa \varepsilon.$$

8 Therefore, we have:

$$\mathcal{A}_{I_1} = \mathbf{Q} + \kappa \widehat{\sigma} \phi \mathbf{I},\tag{21}$$

$$\mathcal{A}_{J_2} = \mathbf{Q} + \kappa \widehat{\sigma} \phi \mathbf{e}_2 \mathbf{e}_2^{\mathsf{T}},\tag{22}$$

$$\mathcal{A}_{J_3} = \mathbf{Q} + \kappa \widehat{\sigma} \phi \mathbf{e}_1 \mathbf{e}_1^{\mathsf{T}},\tag{23}$$

$$\mathcal{A}_{J_4} = Q. \tag{24}$$

Start with \mathcal{A}_{J_4} by expanding **Q**:

$$\begin{split} Q &= (I - \delta K) - \beta (I - \delta K) K - \kappa \widehat{\sigma} K \\ &= (I - \delta K) (I - \beta K) - \kappa \widehat{\sigma} K \\ &= \begin{bmatrix} 1 - \delta p & -\delta (1 - p) \\ -\delta (1 - q) & 1 - \delta q \end{bmatrix} \begin{bmatrix} 1 - \beta p & -\beta (1 - p) \\ -\beta (1 - q) & 1 - \beta q \end{bmatrix} - \kappa \widehat{\sigma} \begin{bmatrix} p & 1 - p \\ 1 - q & q \end{bmatrix} \\ &= \begin{bmatrix} (1 - \delta p) (1 - \beta p) + \delta (1 - p) \beta (1 - q) - \kappa \widehat{\sigma} p & (1 - \delta p) (-\beta (1 - p)) + -\delta (1 - p) (1 - \beta q) - \kappa \widehat{\sigma} (1 - p) \\ -\delta (1 - q) (1 - \beta p) + (1 - \delta q) (-\beta (1 - q)) - \kappa \widehat{\sigma} (1 - q) & -\delta (1 - q) (-\beta (1 - p)) + (1 - \delta q) (1 - \beta q) - \kappa \widehat{\sigma} q \end{bmatrix} \\ &= \begin{bmatrix} (1 - \delta) (1 - \beta p) + \delta (1 - p) [1 - \beta p - \beta q + \beta] - \kappa \widehat{\sigma} p & -\beta (1 - \delta) (1 - p) - \delta (1 - p) [1 - \beta p - \beta q + \beta] - \kappa \widehat{\sigma} (1 - p) \\ -\beta (1 - \delta) (1 - q) - \delta (1 - q) [1 - \beta p - \beta q + \beta] - \kappa \widehat{\sigma} (1 - q) & (1 - \delta) (1 - \beta q) + \delta (1 - q) [1 - \beta p - \beta q + \beta] - \kappa \widehat{\sigma} (1 - q) \end{bmatrix}. \end{split}$$

Thus, we can write \mathcal{A}_{J_1} as:

$$\mathcal{A}_{J_1} = \begin{bmatrix} (1-\delta)(1-\beta p) + \delta(1-p)[1-\beta p - \beta q + \beta] - \kappa \widehat{\sigma} p + \kappa \widehat{\sigma} \phi & -\beta(1-\delta)(1-p) - \delta(1-p)[1-\beta p - \beta q + \beta] - \kappa \widehat{\sigma}(1-p) \\ -\beta(1-\delta)(1-q) - \delta(1-q)[1-\beta p - \beta q + \beta] - \kappa \widehat{\sigma}(1-q) & (1-\delta)(1-\beta q) + \delta(1-q)[1-\beta p - \beta q + \beta] - \kappa \widehat{\sigma} q + \kappa \widehat{\sigma} \phi \end{bmatrix},$$

and so the determinant is

$$\det \mathcal{A}_{J_1} = \left[(1-\delta)(1-\beta p) + \delta(1-p)\left[1-\beta p - \beta q + \beta\right] - \kappa \widehat{\sigma} p + \kappa \widehat{\sigma} \phi \right] \left[(1-\delta)(1-\beta q) + \delta(1-q)\left[1-\beta p - \beta q + \beta\right] - \kappa \widehat{\sigma} q + \kappa \widehat{\sigma} \phi \right] \\ - \left[-\beta(1-\delta)(1-p) - \delta(1-p)\left[1-\beta p - \beta q + \beta\right] - \kappa \widehat{\sigma}(1-p) \right] \left[-\beta(1-\delta)(1-q) - \delta(1-q)\left[1-\beta p - \beta q + \beta\right] - \kappa \widehat{\sigma}(1-q) \right] \\ = a_{1,1} - a_{1,2}.$$

21

The first term on the RHS, $a_{1,1}$,

$$a_{1,1} = \left[(1-\delta)(1-\beta p) + \delta(1-p)[1-\beta p - \beta q + \beta] - \kappa \widehat{\sigma} p + \kappa \widehat{\sigma} \phi \right] \\ \times \left[(1-\delta)(1-\beta q) + \delta(1-q)[1-\beta p - \beta q + \beta] - \kappa \widehat{\sigma} q + \kappa \widehat{\sigma} \phi \right],$$

after much algebraic manipulation can be written as:

$$\begin{split} a_{1,1} &= \left[(1-\delta) \left(1-\beta p\right) + (1-p) \left[\delta \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \right] + \kappa \widehat{\sigma} \left(\phi-1\right) \right] \left[(1-\delta) \left(1-\beta q\right) \right] \\ &+ (1-\delta) \left(1-\beta p\right) \left(1-q\right) \left[\delta \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \right]^2 \\ &+ (1-p) \left(1-q\right) \left[\beta \left(1-\delta\right) \right]^2 \\ &- 2 \left(1-p\right) \left(1-q\right) \left\{ \left[\beta \left(1-\delta\right) \right]^2 + \delta \beta \left(1-\delta\right) \left[1-\beta p-\beta q+\beta\right] + \beta \left(1-\delta\right) \kappa \widehat{\sigma} \right\} \\ &+ \kappa \widehat{\sigma} \left(\phi-1\right) \left(1-q\right) \left[\delta \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \right] \\ &+ \left(1-\delta\right) \left(1-\beta p\right) \kappa \widehat{\sigma} \left(\phi-1\right) \\ &+ \left(1-p\right) \left[\delta \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \right] \kappa \widehat{\sigma} \left(\phi-1\right) \\ &+ \left[\kappa \widehat{\sigma} \left(\phi-1\right) \right]^2. \end{split}$$

The second term is, more simply,

$$a_{1,2} = (1-p)(1-q) \left[\beta (1-\delta) + \delta \left[1-\beta p - \beta q + \beta\right] + \kappa \widehat{\sigma}\right]^2.$$

22

The difference between $a_{1,1}$ and $a_{1,2}$ is:

$$\begin{bmatrix} (1-\delta)(1-\beta p) + (1-p)\left[\delta(1-\beta p - \beta q + \beta) + \kappa \widehat{\sigma}\right] + \kappa \widehat{\sigma} (\phi - 1) \right] (1-\delta)(1-\beta q) + (1-\delta)(1-\beta p)(1-q)\left[\delta(1-\beta p - \beta q + \beta) + \kappa \widehat{\sigma}\right] + (1-p)(1-q)\left[\beta(1-\delta)\right]^{2} + \delta\beta(1-\delta)(1-\beta p - \beta q + \beta) + \beta(1-\delta)\kappa \widehat{\sigma} \right] + \kappa \widehat{\sigma} (\phi - 1)(1-q)\left[\delta(1-\beta p - \beta q + \beta) + \kappa \widehat{\sigma}\right] + (1-\delta)(1-\beta p)\kappa \widehat{\sigma} (\phi - 1) + (1-p)\left[\delta(1-\beta p - \beta q + \beta) + \kappa \widehat{\sigma}\right]\kappa \widehat{\sigma} (\phi - 1) = \kappa \widehat{\sigma} \left[(1-\delta)(1-\beta) + \kappa \widehat{\sigma} (\phi - 1) \right] \left[\frac{\delta(2-p-q)(1-\beta p - \beta q + \beta)}{\kappa \widehat{\sigma}} + 1 - p - q + \frac{(1-\delta)(1-\beta q - \beta p + \beta)}{\kappa \widehat{\sigma}} + \phi \right].$$

23

Then proceed with \mathcal{A}_{J_2} :

$$\begin{aligned} \mathcal{A}_{J_2} &= \mathbf{Q} + \kappa \widehat{\sigma} \phi \mathbf{e}_2 \mathbf{e}_2^\top \\ &= \begin{bmatrix} (1-\delta)(1-\beta p) + \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} p & -\beta(1-\delta)(1-p) - \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-p) \\ -\beta(1-\delta)(1-q) - \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-q) & (1-\delta)(1-\beta q) + \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} q + \kappa \widehat{\sigma} \phi \end{bmatrix}, \end{aligned}$$

where the determinant is:

$$\det \mathcal{A}_{J_2} = \left[(1-\delta)(1-\beta p) + \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} p \right] \left[(1-\delta)(1-\beta q) + \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} q + \kappa \widehat{\sigma} \phi \right] \\ - \left[-\beta(1-\delta)(1-p) - \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-p) \right] \left[-\beta(1-\delta)(1-q) - \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-q) \right] \\ = a_{2,1} - a_{2,2}.$$

Where, after algebraic manipulation we have:

$$\begin{split} a_{2,1} &= (1-\delta)^2 \left(1-\beta p\right) \left(1-\beta q\right) + (1-\delta) \left(1-\beta p\right) \left[\delta \left(1-q\right) \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \left(1-q\right)\right] \\ &+ (1-\delta) \left(1-\beta p\right) \kappa \widehat{\sigma} \left(\phi-1\right) + \left[\delta \left(1-p\right) \left(1-\beta p-\beta q+\beta\right) - \kappa \widehat{\sigma} p\right] \left(1-\delta) \left(1-\beta q\right) \\ &+ \left[\delta \left(1-p\right) \left(1-\beta p-\beta q+\beta\right) - \kappa \widehat{\sigma} p\right] \left[\delta \left(1-q\right) \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \left(1-q\right)\right] \\ &+ \left[\delta \left(1-p\right) \left(1-\beta p-\beta q+\beta\right) - \kappa \widehat{\sigma} p\right] \kappa \widehat{\sigma} \left(\phi-1\right), \end{split}$$

and

$$\begin{split} a_{2,2} &= \beta^2 \left(1-\delta\right)^2 \left(1-p\right) \left(1-q\right) + \left[\beta \left(1-\delta\right) \left(1-p\right)\right] \left[\delta \left(1-q\right) \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \left(1-q\right)\right] \\ &+ \left[\delta \left(1-p\right) \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \left(1-p\right)\right] \beta \left(1-\delta\right) \left(1-q\right) \\ &+ \left[\delta \left(1-p\right) \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \left(1-p\right)\right] \left[\delta \left(1-q\right) \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \left(1-q\right)\right]. \end{split}$$

24

This gives:

$$a_{2,1} - a_{2,2} = \left[\left(\kappa \widehat{\sigma} \right)^2 \left(\phi - 1 \right) + \kappa \widehat{\sigma} \left(1 - \delta \right) \left(1 - \beta \right) \right] \left[\frac{\left(1 - \delta \right) \left(1 - \beta q - \beta p + \beta \right)}{\kappa \widehat{\sigma}} + \frac{\delta \left(2 - p - q \right) \left(1 - \beta p - \beta q + \beta \right)}{\kappa \widehat{\sigma}} + 1 - p - q \right] - \kappa \widehat{\sigma} \phi \left(1 - q \right) \left[\delta \left(1 - \beta p - \beta q + \beta \right) + \kappa \widehat{\sigma} + \beta \left(1 - \delta \right) \right].$$

$$(26)$$

Similarly, for \mathcal{A}_{J_3} :

$$\begin{aligned} \mathcal{A}_{J_3} &= \mathbf{Q} + \kappa \widehat{\sigma} \phi \mathbf{e}_1 \mathbf{e}_1^\top \\ &= \begin{bmatrix} (1-\delta)(1-\beta p) + \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} p + \kappa \widehat{\sigma} \phi & -\beta(1-\delta)(1-p) - \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-p) \\ -\beta(1-\delta)(1-q) - \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-q) & (1-\delta)(1-\beta q) + \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} q \end{bmatrix}. \end{aligned}$$

det \mathcal{A}_{J_3} is:

$$\det \mathcal{A}_{J_3} = \left[(1-\delta)(1-\beta p) + \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} p + \kappa \widehat{\sigma} \phi \right] \left[(1-\delta)(1-\beta q) + \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} q \right] \\ - \left[-\beta(1-\delta)(1-p) - \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-p) \right] \left[-\beta(1-\delta)(1-q) - \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-q) \right] \\ = a_{3,1} - a_{3,2},$$

and – again – after algebraic manipulation:

$$\begin{split} a_{3,1} &= (1-\delta)^2 \left(1-\beta p\right) \left(1-\beta q\right) + (1-\delta) \left(2-p-q-\beta p-\beta q+2\beta pq\right) \delta \left(1-\beta p-\beta q+\beta\right) \\ &+ (1-\delta) \left(1-\beta p\right) \left(-\kappa \widehat{\sigma} q\right) + \delta \left(1-p\right) \left(1-\beta p-\beta q+\beta\right) \\ &- \kappa \widehat{\sigma} \delta \left[(1-p) q+(1-q) p \right] \left(1-\beta p-\beta q+\beta\right) \\ &+ \left(-\kappa \widehat{\sigma} p\right) \left(1-\delta\right) \left(1-\beta q\right) \\ &+ \kappa \widehat{\sigma} \phi \left(1-\delta\right) \left(1-\beta q\right) \\ &+ \kappa \widehat{\sigma} \phi \left(1-\delta\right) \left(1-\beta p-\beta q+\beta\right) \\ &+ \kappa \widehat{\sigma} \phi \left(-\kappa \widehat{\sigma} q\right), \end{split}$$

and

$$\begin{aligned} a_{3,2} &= \left[-\beta \left(1-\delta \right) \left(1-p \right) - \delta \left(1-p \right) \left(1-\beta p-\beta q+\beta \right) - \kappa \widehat{\sigma} \left(1-p \right) \right] \left[-\beta \left(1-\delta \right) \left(1-q \right) - \delta \left(1-q \right) \left(1-\beta p-\beta q+\beta \right) - \kappa \widehat{\sigma} \left(1-q \right) \right] \right] \\ &= \beta^2 \left(1-\delta \right)^2 \left(1-p \right) \left(1-q \right) + \beta \left(1-\delta \right) \left(1-p \right) \delta \left(1-q \right) \left(1-\beta p-\beta q+\beta \right) + \beta \left(1-\delta \right) \left(1-p \right) \kappa \widehat{\sigma} \left(1-q \right) \\ &+ \beta \left(1-\delta \right) \left(1-q \right) \delta \left(1-p \right) \left(1-\beta p-\beta q+\beta \right) + \delta \left(1-p \right) \left(1-\beta p-\beta q+\beta \right) \delta \left(1-q \right) \left(1-\beta p-\beta q+\beta \right) \\ &+ \delta \left(1-p \right) \left(1-\beta p-\beta q+\beta \right) \kappa \widehat{\sigma} \left(1-q \right) + \kappa \widehat{\sigma} \left(1-p \right) \left[\beta \left(1-\delta \right) \left(1-q \right) + \delta \left(1-q \right) \left(1-\beta p-\beta q+\beta \right) + \kappa \widehat{\sigma} \left(1-q \right) \right] \end{aligned}$$

$$a_{3,1} - a_{3,2} = \kappa \widehat{\sigma} \left(1 - \delta\right) \left(1 - \beta\right) - \left(\kappa \widehat{\sigma}\right)^2 \left[\frac{\left(1 - \delta\right) \left(1 - \beta q - \beta p + \beta\right)}{\kappa \widehat{\sigma}} + \frac{\delta \left(2 - p - q\right) \left(1 - \beta p - \beta q + \beta\right)}{\kappa \widehat{\sigma}} + 1 - p - q + \phi \right] + \kappa \widehat{\sigma} \phi \left(1 - q\right) \left[\beta \left(1 - \delta\right) + \delta \left(1 - \beta p - \beta q + \beta\right) + \kappa \widehat{\sigma} \right].$$

$$(27)$$

Finally, go back and use Q to write \mathcal{R}_{J_4} :

 $\mathcal{A}_{J_4}=Q,$

with

$$\det \mathcal{A}_{J_4} = \left[(1-\delta)(1-\beta p) + \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} p \right] \left[(1-\delta)(1-\beta q) + \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma} q \right] \\ - \left[-\beta(1-\delta)(1-p) - \delta(1-p)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-p) \right] \left[-\beta(1-\delta)(1-q) - \delta(1-q)(1-\beta p - \beta q + \beta) - \kappa \widehat{\sigma}(1-q) \right] \\ = a_{4,1} - a_{4,2}.$$

The first term $a_{4,1}$ is:

26

$$\begin{split} a_{4,1} &= (1-\delta)\left(1-\beta p\right)\left(1-\delta\right)\left(1-\beta q\right) + (1-\delta)\left(1-\beta p\right)\left[\delta\left(1-q\right)\left(1-\beta p-\beta q+\beta\right) + \kappa\widehat{\sigma}\left(1-q\right)\right] \\ &- (1-\delta)\left(1-\beta p\right)\kappa\widehat{\sigma} + (1-\delta)\left(1-\beta q\right)\left[\delta\left(1-p\right)\left(1-\beta p-\beta q+\beta\right) + \kappa\widehat{\sigma}\left(1-p\right)\right] - (1-\delta)\left(1-\beta q\right)\kappa\widehat{\sigma} \\ &+ \delta\left(1-p\right)\left[1-\beta p-\beta q+\beta\right] - \kappa\widehat{\sigma}p\left[\delta\left(1-q\right)\left(1-\beta p-\beta q+\beta\right) - \kappa\widehat{\sigma}q\right], \end{split}$$

and the second term $a_{4,2}$ is:

$$\begin{aligned} a_{4,2} &= \beta \left(1-\delta\right) \left(1-p\right) \beta \left(1-\delta\right) \left(1-q\right) + 2\beta \left(1-\delta\right) \left(1-q\right) \left(1-p\right) \left[\delta \left(1-\beta p-\beta q+\beta\right)+\kappa \widehat{\sigma}\right] \\ &+ \left(1-q\right) \left(1-p\right) \left[\delta \left(1-\beta p-\beta q+\beta\right)+\kappa \widehat{\sigma}\right]^2. \end{aligned}$$

With

$$a_{4,1} - a_{4,2} = \left[-\left(\kappa\widehat{\sigma}\right)^2 + \kappa\widehat{\sigma}\left(1 - \delta\right)\left(1 - \beta\right) \right] \left[\frac{\left(1 - \delta\right)\left(1 - \beta p - \beta q + \beta\right)}{\kappa\widehat{\sigma}} + \frac{\delta\left(2 - q - p\right)\left(1 - \beta p - \beta q + \beta\right)}{\kappa\widehat{\sigma}} + 1 - q - p \right].$$
(28)

Given the expressions in (18), we only need to ensure that the signs of det \mathcal{A}_{J_1} and det \mathcal{A}_{J_4} are identical:

$$\begin{aligned} \det \boldsymbol{\mathcal{A}}_{J_1} &= \kappa \widehat{\sigma} \left[(1-\delta) \left(1-\beta\right) + \kappa \widehat{\sigma} \left(\phi-1\right) \right] \left(\phi-\vartheta\right), \\ \det \boldsymbol{\mathcal{A}}_{J_2} &= \left[\left(\kappa \widehat{\sigma}\right)^2 \left(\phi-1\right) + \kappa \widehat{\sigma} \left(1-\delta\right) \left(1-\beta\right) \right] \left(-\vartheta\right) \\ &- \kappa \widehat{\sigma} \phi \left(1-q\right) \left[\delta \left[1-\beta p-\beta q+\beta\right] + \kappa \widehat{\sigma} + \beta \left(1-\delta\right) \right], \\ \det \boldsymbol{\mathcal{A}}_{J_3} &= \left[\kappa \widehat{\sigma} \left[\left(1-\delta\right) \left(1-\beta\right) \right] - \left(\kappa \widehat{\sigma}\right)^2 \right] \left(\phi-\vartheta\right) \\ &+ \kappa \widehat{\sigma} \phi \left(1-q\right) \left[\beta \left(1-\delta\right) + \delta \left(1-\beta p-\beta q+\beta\right) + \kappa \widehat{\sigma} \right], \\ \det \boldsymbol{\mathcal{A}}_{J_4} &= \left[- \left(\kappa \widehat{\sigma}\right)^2 + \kappa \widehat{\sigma} \left(1-\delta\right) \left(1-\beta\right) \right] \left(-\vartheta\right), \end{aligned}$$

where

$$\vartheta \equiv p + q - 1 - \frac{\left[(1 - \delta) + \delta \left(2 - p - q\right)\right] \left(1 - \beta q - \beta p + \beta\right)}{\kappa \widehat{\sigma}}$$

det \mathcal{A}_{J_1} is trivially positive given $\hat{\sigma} < 0$ and $\phi < \vartheta$. Similarly det \mathcal{A}_{J_4} is trivially positive.

When q = 1, the determinants simplify to

$$\det \mathcal{A}_{J_1} = \left[\left(\kappa \widehat{\sigma} \right)^2 \left(\phi - 1 \right) + \kappa \widehat{\sigma} \left(1 - \delta \right) \left(1 - \beta \right) \right] \left(\phi - \vartheta \right), \tag{29}$$

$$\det \mathcal{A}_{J_2} = \left[\left(\kappa \widehat{\sigma} \right)^2 \left(\phi - 1 \right) + \kappa \widehat{\sigma} \left(1 - \delta \right) \left(1 - \beta \right) \right] (-\vartheta) , \qquad (30)$$

$$\det \mathcal{A}_{J_3} = \left[\kappa \widehat{\sigma} \left[(1 - \delta) \left(1 - \beta \right) \right] - \left(\kappa \widehat{\sigma} \right)^2 \right] \left(\phi - \vartheta \right), \tag{31}$$

$$\det \mathcal{A}_{J_4} = \left[-\left(\kappa \widehat{\sigma}\right)^2 + \kappa \widehat{\sigma} \left(1 - \delta\right) \left(1 - \beta\right) \right] \left(-\vartheta\right), \tag{32}$$

and

$$\vartheta \equiv p - \frac{\left(1 - \delta p\right)\left(1 - \beta p\right)}{\kappa \widehat{\sigma}}$$

As documented in AM, for standard calibrations of the RANK model, these determinants have different signs and thus the RANK model, when subject to an occasionally binding ELB constraint, generally does not possess a unique MSV solution. In the language of Gourieroux, Laffont, and Monfort (1980) and AM, this is referred to as a violation of coherency and completeness. For simplicity, we use the terms existence and uniqueness instead of coherency and completeness, respectively. **Proof for Proposition 1.** To check the signs for the HANK model, and with δ and $\hat{\sigma}$ given in (4), we have to cover three cases:

- 1. When $\hat{\sigma} < 0$, we have $1 \lambda \chi < 0$ and thus $\chi > \frac{1}{\lambda} > 1$, which implies that $\delta < 1$.
- 2. When $\widehat{\sigma} > 0$, $\chi < \frac{1}{\lambda}$. If $1 < \chi < \frac{1}{\lambda}$, then $\delta > 1$.
- 3. When $\widehat{\sigma} > 0$, $\chi < \frac{1}{\lambda}$. If $\chi < 1$, then $\delta < 1$.

Case 1 ($\hat{\sigma} < 0$ and $\delta < 1$): If $\hat{\sigma} < 0$, this implies $\delta < 1$, and so

$$\vartheta \equiv p - \frac{(1 - \delta p) (1 - \beta p)}{\kappa \widehat{\sigma}} > p.$$

This implies that

$$\det \mathcal{A}_{J_4} = \underbrace{\left[-\left(\kappa\widehat{\sigma}\right)^2 + \kappa\widehat{\sigma}\left(1-\delta\right)\left(1-\beta\right)\right]}_{<0}\underbrace{\left(-\vartheta\right)}_{<0} > 0.$$

Then, for det \mathcal{A}_{J_3} to be positive, we require

$$\phi - \vartheta < 0 \implies \delta p < 1 + \frac{\kappa \widehat{\sigma} (\phi - p)}{1 - \beta p}.$$

This implies that, for det \mathcal{A}_{J_1} to be positive, we need

$$\begin{split} \left(\kappa\widehat{\sigma}\right)^{2}\left(\phi-1\right)+\kappa\widehat{\sigma}\left(1-\delta\right)\left(1-\beta\right)<0,\\ \Longrightarrow \ \frac{\kappa\widehat{\sigma}\left(\phi-1\right)}{1-\beta}+1>\delta. \end{split}$$

This implies that:

$$\det \mathcal{A}_{J_2} = \underbrace{\left[\left(\kappa \widehat{\sigma}\right)^2 \left(\phi - 1\right) + \kappa \widehat{\sigma} \left(1 - \delta\right) \left(1 - \beta\right)\right]}_{<0} \underbrace{\left(-\vartheta\right)}_{<0} > 0.$$

Case 2 ($\widehat{\sigma} > 0$, $1 < \chi < \frac{1}{\lambda}$, and $\delta > 1$): If $\widehat{\sigma} > 0$ and $\delta < 1$, and also suppose $\phi > 1$, then:

$$\vartheta \equiv p - \frac{(1 - \delta p)(1 - \beta p)}{\kappa \widehat{\sigma}} < p.$$

That implies

$$\det \mathcal{A}_{J_1} = \underbrace{\left[\left(\kappa \widehat{\sigma}\right)^2 \left(\phi - 1\right) + \kappa \widehat{\sigma} \left(1 - \delta\right) \left(1 - \beta\right)\right]}_{>0} \underbrace{\left(\phi - \vartheta\right)}_{>0} > 0.$$

For det \mathcal{A}_{J_2} to be positive we require $-\vartheta > 0$, i.e., $\vartheta < 0$, which then implies:

$$\implies \delta p < 1 - \frac{\kappa \widehat{\sigma} p}{1 - \beta p}.$$

This implies that

det
$$\mathcal{A}_{J_4} = \left[-\left(\kappa \widehat{\sigma}\right)^2 + \kappa \widehat{\sigma} \left(1 - \delta\right) \left(1 - \beta\right) \right] \underbrace{\left(-\vartheta\right)}_{>0}$$
.

We need

$$-(\kappa\widehat{\sigma})^{2} + \kappa\widehat{\sigma}(1-\delta)(1-\beta) > 0,$$
$$\implies 1 - \frac{\kappa\widehat{\sigma}}{(1-\beta)} > \delta.$$

Then

$$\det \mathcal{A}_{J_3} = \left[-\left(\kappa \widehat{\sigma}\right)^2 + \kappa \widehat{\sigma} \left(1 - \delta\right) \left(1 - \beta\right) \right] \underbrace{(\phi - \vartheta)}_{>0} > 0, \quad \text{if} \quad 1 - \frac{\kappa \widehat{\sigma}}{(1 - \beta)} > \delta.$$

Case 3 ($\hat{\sigma} > 0, \chi < 1$, and $\delta < 1$): If $\hat{\sigma} > 0$ and $\delta > 1$ with $\phi > 1$, then we cannot conclude anything about about ϑ , and so the signs of det \mathcal{A}_{J_3} and det \mathcal{A}_{J_4} are ambiguous,

$$\det \mathcal{A}_{J_3} = \underbrace{\left[-\left(\kappa\widehat{\sigma}\right)^2 + \kappa\widehat{\sigma}\left(1-\delta\right)\left(1-\beta\right)\right]}_{<0} \left[\phi-\vartheta\right],$$
$$\det \mathcal{A}_{J_4} = \underbrace{\left[-\left(\kappa\widehat{\sigma}\right)^2 + \kappa\widehat{\sigma}\left(1-\delta\right)\left(1-\beta\right)\right]}_{<0} \left[-\vartheta\right].$$

We need $\phi - \vartheta$ and $-\vartheta$ to have the same sign. First, suppose they are both positive $(-\vartheta > 0)$ which implies

$$\delta p < 1 - \frac{\kappa \widehat{\sigma} p}{(1 - \beta p)}$$

and

$$\delta p < 1 + \frac{\kappa \widehat{\sigma} (\phi - p)}{(1 - \beta p)}.$$

Then det \mathcal{A}_{J_3} and det \mathcal{A}_{J_4} are negative. For det \mathcal{A}_{J_1} and det \mathcal{A}_{J_2} to both be negative, we then need

$$\begin{split} \left[\left(\kappa \widehat{\sigma} \right)^2 \left(\phi - 1 \right) + \kappa \widehat{\sigma} \left(1 - \delta \right) \left(1 - \beta \right) \right] &< 0, \\ \implies 1 + \frac{\kappa \widehat{\sigma} \left(\phi - 1 \right)}{\left(1 - \beta \right)} &< \delta. \end{split}$$

But this holds only for a sufficiently small *p*.

Secondly, suppose both $\phi - \vartheta$ and $-\vartheta$ are negative,

$$\begin{split} \phi - \vartheta &< 0, \\ \Longrightarrow \ p - \frac{(1 - \delta p)(1 - \beta p)}{\kappa \widehat{\sigma}} > \phi, \\ \implies \delta p > 1 + \frac{\kappa \widehat{\sigma} (\phi - p)}{(1 - \beta p)} \end{split}$$

Then we require

$$(\kappa \widehat{\sigma})^2 (\phi - 1) + \kappa \widehat{\sigma} (1 - \delta) (1 - \beta) > 0,$$
$$\implies 1 + \frac{\kappa \widehat{\sigma} (\phi - 1)}{(1 - \beta)} > \delta.$$

For the first case, compare

$$\delta p < 1 - \frac{\kappa \widehat{\sigma} p}{(1 - \beta p)}$$
, and $p + \frac{p \kappa \widehat{\sigma} (\phi - 1)}{(1 - \beta)} .$

We need

$$1 - \frac{\kappa \widehat{\sigma} p}{(1 - \beta p)} > p + \frac{p \kappa \widehat{\sigma} (\phi - 1)}{(1 - \beta)}$$

which is satisfied for small values of $p \in [0, p^*]$. For the second case, compare

$$\delta p > 1 + \frac{\kappa \widehat{\sigma} (\phi - p)}{(1 - \beta p)}, \text{ and } p + \frac{p \kappa \widehat{\sigma} (\phi - 1)}{(1 - \beta)} > p \delta.$$

We need

$$p + \frac{p\kappa\widehat{\sigma}(\phi - 1)}{(1 - \beta)} > 1 + \frac{\kappa\widehat{\sigma}(\phi - p)}{(1 - \beta p)},$$

which is satisfied for large values of $p \in [p^*, 1]$.

A.2 Robust real rate rule

In order to write the unconstrained model in terms of $\{x_t, \pi_t\}$, we use the NKPC (2) and (13) (with the Fisher equation) to write:

$$\pi_t = \frac{\kappa}{1 - \beta \phi} x_t,\tag{33}$$

$$\pi_t = \frac{1}{\phi} \mathbb{E}_t \pi_{t+1}. \tag{34}$$

However, in the ELB-constrained $(i_t = -\mu)$ regime we have:

$$x_t = \delta \mathbb{E}_t x_{t+1} + \widehat{\sigma}(\mu + \mathbb{E}_t \pi_{t+1}) + \varepsilon_t, \qquad (35)$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t. \tag{36}$$

As before, we can write the system for the unconstrained regime in canonical form (15) with:

$$\underbrace{\begin{bmatrix} \frac{\kappa}{\beta\phi-1} & 1\\ 0 & 1 \end{bmatrix}}_{A_1} \begin{bmatrix} x_t\\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0\\ 0 & -\frac{1}{\phi} \end{bmatrix}}_{B_1} \begin{bmatrix} \mathbb{E}_t x_{t+1}\\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0\\ 0 & 0 \end{bmatrix}}_{C_1} \begin{bmatrix} \varepsilon_t\\ \mu \end{bmatrix} = \mathbf{0}, \quad \text{for } i_t > -\mu.$$
(37)

and the matrices for the constrained regime $\{A_0, B_0, C_0\}$ are identical to those in (17).

As in Appendix A.1, we calculate the analytical expressions for the determinants of matrices \mathcal{A}_{J_1} , \mathcal{A}_{J_2} , \mathcal{A}_{J_3} , and \mathcal{A}_{J_4} :

$$\det \mathcal{A}_{J_1} = \frac{\kappa}{\beta\phi - 1} \left(1 - \frac{q}{\phi} \right) + \frac{1 - q}{\phi},$$

$$\det \mathcal{A}_{J_2} = \frac{\kappa}{\beta\phi - 1} (1 - \beta q) + \kappa + \beta(1 - q),$$

$$\det \mathcal{A}_{J_3} = \left[1 - \delta p - \widehat{\sigma}(1 - q) \right] \left(1 - \frac{q}{\phi} \right) + \left[\delta(1 - p) + \widehat{\sigma}q \right] \left(\frac{1 - q}{\phi} \right),$$

$$\det \mathcal{A}_{J_4} = \left[-(\kappa \widehat{\sigma})^2 + \kappa \widehat{\sigma}(1 - \delta)(1 - \beta) \right] (-\vartheta).$$

In A.1, we have shown that det \mathcal{A}_{J_4} , which is equivalent across the two cases, is positive. For $\phi > 1$, det \mathcal{A}_{J_1} is also positive.

Global indeterminacy even with a RR rule. Recall that in order for a unique MSV solution to exist in a forward-looking dynamic model, the signs of the determinants of (18) must all share the same sign. Given our canonical form matrices (17) and (37), this is straightforward to analytically verify. Moreover, as before, we only need to ensure that the signs of det \mathcal{A}_{J_1} and det \mathcal{A}_{J_4} are the same. We also already know det \mathcal{A}_{J_4} since the ELB-constrained regimes with the Taylor-type rule (3) and RR rule (13) are identical.

So, constructing \mathcal{A}_{I_1} from (37) and finding the determinant yields

$$\det \mathcal{A}_{J_1} = \begin{vmatrix} \frac{\kappa}{\beta\phi-1} & 1\\ -\frac{(1-q)}{\phi} & 1-\frac{q}{\phi} \end{vmatrix} = \frac{\kappa}{\beta\phi-1} \left(1-\frac{q}{\phi}\right) - \frac{q-1}{\phi}.$$

To simplify the analysis, assume that p = q = 1, which also allows us to reuse our result

from (32). We thus compare the signs of:

$$\det \mathcal{A}_{J_1} = \frac{\kappa(\phi - 1)}{\beta \phi^2 - \phi},$$
$$\det \mathcal{A}_{J_4} = (\delta - 1)(\beta - 1) - \widehat{\sigma}\kappa$$

Since $\phi > 1$, we have that det $\mathcal{A}_{J_1} > 0$. However, there appears to be a tension between the sign of det \mathcal{A}_{J_4} and the class of HANK model one assumes, much like with a Taylortype rule. If we assume a HANK-C model, then det $\mathcal{A}_{J_4} < 0$ since $\delta > 1$, and so the model is not globally determinate. However, with a HANK-P model, in order for det $\mathcal{A}_{J_4} > 0$, we would need to satisfy the same condition as in (8). Recall that such a condition required a high level of risk aversion, or equivalently, a very small IES – neither of which are supported by empirical evidence.