Online Appendix for "Productivity over the Life-Cycle and its Effects on the Interest Rate"

Momo Komatsu, David Murakami, and Ivan Shchapov

A Model solutions

A.1 The elder worker problem

Using the definition of financial assets, A_t, an elderly individual wishes to maximise (5) subject to (6). Their problem can be written recursively as:

$$V_t^e(j,k) = \max_{C_t^e(j,k), L_t^e(j,k), A_t^r(j,k)} \left\{ \left(C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon} \right)^{\rho} + \beta \gamma_{t+1} V_{t+1}^e(j,k)^{\rho} \right\}^{\frac{1}{\rho}}$$
$$+ \lambda_t \left\{ \frac{R_{t-1}}{\gamma_t \pi_t} A_{t-1}^e(j,k) + \varsigma_t w_t L_t^e(j,k) + E_t^e(j,k) - C_t^e(j,k) - A_t^e(j,k) \right\}.$$

The first-order conditions with respect to consumption, labour, and financial assets are:

$$\frac{\partial V_t^e(j,k)}{\partial C_t^e(j,k)} = \frac{1}{\rho} \underbrace{\left\{ \left(C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon} \right)^{\rho} + \beta \gamma_{t+1} V_{t+1}^e(j,k)^{\rho} \right\}^{\frac{1}{\rho}-1}}_{V_t^e(j,k)^{1-\rho}}$$
$$\times \rho \left(C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon} \right)^{\rho-1} \upsilon C_t^e(j,k)^{\upsilon-1} \left[1 - L_t^e(j,k)\right]^{1-\upsilon}$$
$$- \lambda_t = 0$$
$$\Leftrightarrow \lambda_t = \upsilon V_t^e(j,k)^{1-\rho} C_t^e(j,k)^{\upsilon\rho-1} \left[1 - L_t^e(j,k)\right]^{\rho(1-\upsilon)}, \tag{A.1}$$
$$\frac{\partial V_t^e(j,k)}{\partial L_t^e(j,k)} = \frac{1}{\rho} V_t^e(j,k)^{1-\rho} \rho \left(C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon} \right)^{\rho-1}$$
$$\times (1-\upsilon) C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon-1} (-1) + \lambda_t \varsigma_t w_t = 0$$
$$\Leftrightarrow \lambda_t \varsigma_t w_t = (1-\upsilon) V_t^e(j,k)^{1-\rho} C_t^e(j,k)^{\upsilon\rho} \left[1 - L_t^e(j,k)\right]^{\rho(1-\upsilon)-1}, \tag{A.2}$$
$$\frac{\partial V_t^e(j,k)}{\partial A_t^e(j,k)} = \frac{1}{\rho} V_t^e(j,k)^{1-\rho} \rho \beta \gamma_{t+1} V_{t+1}^e(j,k)^{\rho-1} \left[\frac{\partial V_{t+1}^e(j,k)}{\partial A_t^e(j,k)} \right] - \lambda_t$$
$$\Leftrightarrow \lambda_t = V_t^e(j,k)^{1-\rho} \beta \gamma_{t+1} V_{t+1}^e(j,k)^{\rho-1} \left[\frac{\partial V_{t+1}^e(j,k)}{\partial A_t^e(j,k)} \right]. \tag{A.3}$$

Combine Equations (A.1) and (A.2) to get the intratemporal Euler equation (7) in the text:

$$1 - L_t^e(j,k) = \frac{1-\upsilon}{\upsilon} \frac{C_t^e(j,k)}{\varsigma_t w_t}.$$

Then use (A.1) and (A.3) to get:

$$\upsilon V_t^e(j,k)^{1-\rho} C_t^e(j,k)^{\upsilon\rho-1}(1-L_t^e(j,k))^{\rho(1-\upsilon)} \\ = V_t^e(j,k)^{1-\rho}\beta\gamma_{t+1}V_{t+1}^e(j,k)^{\rho-1}\left[\frac{\partial V_{t+1}^e(j,k)}{\partial A_t^e(j,k)}\right]. \tag{A.4}$$

By the Envelope Theorem we have:

$$\frac{\partial V_t^e(j,k)}{\partial A_{t-1}^e(j,k)} = \frac{\lambda_t}{\gamma_t}\frac{R_{t-1}}{\pi_t},$$

and then use (A.1) to write:

$$\frac{\partial V_t^e(j,k)}{\partial A_{t-1}^e(j,k)} = \upsilon V_t^e(j,k)^{1-\rho} C_t^e(j,k)^{\upsilon\rho-1}\left[1-L_t^e(j,k)\right]^{\rho(1-\upsilon)}\frac{R_{t-1}}{\gamma_t\pi_t}$$

and roll forward by one period to get:

$$\frac{\partial V_{t+1}^e(j,k)}{\partial A_t^e(j,k)} = \upsilon V_{t+1}^e(j,k)^{1-\rho} C_{t+1}^e(j,k)^{\upsilon\rho-1}\left[1-L_{t+1}^e(j,k)\right]^{\rho(1-\upsilon)}\frac{R_t}{\gamma_{t+1}\pi_{t+1}}.$$

Substitute this value back into (A.4) and do some simplification to yield an intertemporal consumption Euler equation:

$$1 = \beta\frac{R_t}{\pi_{t+1}}\left[\frac{C_{t+1}^e(j,k)}{C_t^e(j,k)}\right]^{\upsilon\rho-1}\left[\frac{1-L_{t+1}^e(j,k)}{1-L_t^e(j,k)}\right]^{\rho(1-\upsilon)}. \tag{A.5}$$

Then use (7) to get:

$$C_{t+1}^e(j,k) = \left(\beta\frac{R_t}{\pi_{t+1}}\right)^{\sigma}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma} C_t^e(j,k),$$

which is Equation (8) in the text and where $\sigma = 1/(1-\rho)$.

To get the law of motion for the MPC of the elderly, start by substituting the guessed consumption function (11) into the intertemporal Euler equation (8):

$$\xi_{t+1}^e\left[\frac{R_t}{\gamma_{t+1}\pi_{t+1}}A_t^e(j,k) + H_{t+1}^e(j,k) + S_{t+1}^e(j,k)\right] \\ = \left(\beta\frac{R_t}{\pi_{t+1}}\right)^{\sigma}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}\xi_t^e\left[\frac{R_{t-1}}{\gamma_t\pi_t}A_{t-1}^e(j,k) + H_t^e(j,k) + S_t^e(j,k)\right], \tag{A.6}$$

Then substitute the guessed consumption function (11) into the budget constraint (6) to get an expression for the dynamics of financial assets of a retiree:

$$A_t^e(j,k) = (1-\xi_t^e)\left[\frac{R_{t-1}}{\gamma_t\pi_t}A_{t-1}^e(j,k) + H_t^e(j,k) + S_t^e(j,k) - \mathcal{T}_t\right]. \tag{A.7}$$

Use this expression, and Equations (9) and (10), and substitute into (A.6), and then rearrange to write:

$$\xi^e_{t+1}\frac{A^e_{t+1}(j,k)}{1-\xi^e_{t+1}} = \left(\frac{\beta R_t}{\pi_{t+1}}\right)^{\sigma}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}\xi^e_t\frac{A_t}{1-\xi^e_t}$$
$$\Leftrightarrow \frac{1}{\xi^e_t} = 1+\left(\frac{\beta R_t}{\pi_{t+1}}\right)^{\sigma}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}\frac{1}{\xi^e_{t+1}} \times \frac{A^e_t(j,k)}{\frac{R_t}{\gamma_{t+1}\pi_{t+1}}A^e_t(j,k)+H^e_{t+1}(j,k)+S^e_{t+1}(j,k)}$$
$$\Leftrightarrow \frac{1}{\xi^e_t} = 1+\gamma_{t+1}\beta^{\sigma}\left(\frac{R_t}{\Pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}\frac{1}{\xi^e_{t+1}} \times \frac{A^e_t(j,k)}{A^e_t(j,k)+H^e_t(j,k)-\varsigma_t w_t L^e_t(j,k)+S^e_t(j,k)-E^e_t(j,k)},$$

and since

$$\varsigma_t w_t L^e_t(j,k)+E^e_t(j,k) = C^e_t(j,k)+A^e_t(j,k)-\frac{R_{t-1}}{\gamma_t\pi_t}A^e_{t-1}(j,k),$$

the denominator of the last term in the $1/\xi^e_t$ expression can be written as:

$$H^e_t(j,k)+S^e_t(j,k)-C^e_t(j,k)+\frac{R_{t-1}}{\gamma_t\pi_t}A^e_{t-1}(j,k).$$

Then use (11) to write this expression as:

$$\frac{R_{t-1}}{\gamma_t\pi_t}A^e_{t-1}(j,k)+H^e_t(j,k)+S^e_t(j,k)-\xi^e_t\left[\frac{R_{t-1}}{\gamma_t\pi_t}A^e_{t-1}(j,k)+H^e_t(j,k)+S^e_t(j,k)\right],$$

and then following (A.7) the above expression is merely $A^e_t(j,k)$, which allows us to write:

$$\frac{1}{\xi^e_t} = 1+\gamma_{t+1}\beta^{\sigma}\left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}\frac{1}{\xi^e_{t+1}},$$

which is Equation (12) in the text.

Next, guess that the value function is linear in consumption and leisure:

$$\begin{aligned} V^e_t(j,k) &= \Lambda^e_t C^e_t(j,k)^{\upsilon}(1-L^e_t(j,k))^{1-\upsilon} \\ &= \Lambda^e_t C^e_t(j,k)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{\varsigma_t w_t}\right)^{1-\upsilon}, \end{aligned} \tag{A.8}$$

where we used (7) for the second line. From the value function (5), we can then write:

$$\left[\Lambda^e_t C^e_t(j,k)\left(\frac{1}{\varsigma_t w_t}\right)^{1-\upsilon}\right]^{\rho} = \left[C^e_t(j,k)\left(\frac{1}{\varsigma_t w_t}\right)^{1-\upsilon}\right]^{\rho} + \beta\gamma_{t+1}\left[\Lambda^e_{t+1}C^e_{t+1}(j,k)\left(\frac{1}{\varsigma_t w_{t+1}}\right)^{1-\upsilon}\right]^{\rho}.$$

Use the expression for C_{t+1} from the intertemporal consumption Euler equation (A.5) and then simplify to get:

$$(\Lambda_t^e)^\rho = 1 + \beta^\sigma \gamma_{t+1} \left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1} \left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-v)\sigma} (\Lambda_{t+1}^e)^\rho.$$

From (12) we can then deduce that:

$$\Lambda_t^e = (\xi_t^e)^{\frac{\sigma}{1-\sigma}}, \tag{A.9}$$

and hence we can get Equation (13).

A.2 The young worker problem

The derivation for the young worker's problem closely follows that of the elder worker in Section A.1. A young worker wishes to maximise (14) subject to (15):

$$\begin{aligned} V_t^y(j) = \max_{C_t^y(j), L_t^y(j), A_t^y(j)} & \left\{ \begin{matrix} \left(C_t^y(j)^v (1-L_t^y(j))^{1-v}\right)^\rho \\ +\beta[\omega_{t+1} V_{t+1}^y(j) + (1-\omega_{t+1}) V_{t+1}^e(j, t+1)]^\rho \end{matrix} \right\}^{\frac{1}{\rho}} \\ & + \lambda_t \left\{ \frac{R_{t-1}}{\pi_t} A_{t-1}^y(j) + w_t L_t^y(j) - C_t^y(j) - A_t^y(j) - T_t^y(j) \right\}. \end{aligned}$$

The first-order conditions with respect to consumption, labour, and financial assets are:

$$\frac{\partial V_t^y(j)}{\partial C_t^y(j)} = v V_t^y(j)^{1-\rho} C_t^y(j)^{v\rho-1} \left[1 - L_t^y(j)\right]^{\rho(1-v)} - \lambda_t = 0, \tag{A.10}$$

$$\frac{\partial V_t^y(j)}{\partial L_t^y(j)} = -(1-v) V_t^y(j)^{1-\rho} C_t^y(j)^{v\rho} (1 - L_t^y(j))^{\rho(1-v)-1} + \lambda_t w_t = 0, \tag{A.11}$$

$$\begin{aligned} \frac{\partial V_t^y(j)}{\partial A_t^y(j)} &= V_t^y(j)^{1-\rho} \beta \left[\omega_{t+1} V_{t+1}^y(j) + (1-\omega_{t+1}) V_{t+1}^e(j, t+1)\right]^{\rho-1} \\ &\times \left[\omega_{t+1} \frac{\partial V_{t+1}^y(j)}{\partial A_t^y(j)} + (1-\omega_{t+1}) \frac{\partial V_{t+1}^e(j, t+1)}{\partial A_t^y(j)}\right] - \lambda_t = 0. \end{aligned} \tag{A.12}$$

Combine (A.10) and (A.11) to get the intratemporal Euler equation:

$$1 - L_t^y(j) = \frac{1-v}{v} \frac{C_t^y(j)}{w_t},$$

which is Equation (16) in the text.

Then use (A.10) and (A.12) to write:

$$\begin{aligned} v V_t^y(j)^{1-\rho} C_t^y(j)^{v\rho-1} (1 - L_t^y(j))^{\rho(1-v} &= V_t^y(j)^{1-\rho} \beta \left[\omega_{t+1} V_{t+1}^y(j) + (1-\omega_{t+1}) V_{t+1}^e(j, t+1)\right]^{\rho-1} \\ &\times \left[\omega_{t+1} \frac{\partial V_{t+1}^y(j)}{\partial A_t^y(j)} + (1-\omega_{t+1}) \frac{\partial V_{t+1}^e(j, t+1)}{\partial A_t^y(j)}\right]. \end{aligned}$$

The envelope conditions are:

$$\frac{\partial V_{t+1}^{y}(j)}{\partial A_{t}^{y}(j)} = \upsilon V_{t+1}^{y}(j)^{1-\rho} C_{t+1}^{y}(j)^{\upsilon\rho-1}(1-L_{t+1}^{y}(j))^{\rho(1-\upsilon)} \frac{R_t}{\pi_{t+1}},$$

$$\frac{\partial V_{t+1}^{e}(j,t+1)}{\partial A_{t}^{y}(j)} = \upsilon V_{t+1}^{e}(j,t+1)^{1-\rho} C_{t+1}^{e}(j,t+1)^{\upsilon\rho-1}(1-L_{t+1}^{e}(j,t+1))^{\rho(1-\upsilon)} \frac{R_t}{\pi_{t+1}}.$$

Combining the above envelope conditions with (A.10) and (A.12) yields the following:

$$C_t^y(j)^{\upsilon\rho-1}\left[1-L_t^y(j)\right]^{\rho(1-\upsilon)} = \frac{\beta R_t}{\pi_{t+1}} \left\{ \begin{array}{c} \left[\omega_{t+1}V_{t+1}^{y}(j) + (1-\omega_{t+1})V_{t+1}^{e}(j,t+1)\right]^{\rho-1} \\ \times\omega_{t+1}V_{t+1}^{y}(j)^{1-\rho}C_{t+1}^{y}(j)^{\upsilon\rho-1}\left[1-L_{t+1}^{y}(j)\right]^{\rho(1-\upsilon)} \\ +\left[\omega_{t+1}V_{t+1}^{y}(j) + (1-\omega_{t+1})V_{t+1}^{e}(j,t+1)\right]^{\rho-1} \\ \times(1-\omega_{t+1})V_{t+1}^{e}(j,t+1)^{1-\rho} \\ \times C_{t+1}^{e}(j,t+1)^{\upsilon\rho-1}\left[1-L_{t+1}^{e}(j,t+1)\right]^{\rho(1-\upsilon)} \end{array} \right\}.$$

Then, based on (A.8), (7), and (16), conjecture that the value functions are linear in consumption and leisure:

$$V_t^y(j) = \Lambda_t^y C_t^y(j)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{w_t}\right)^{1-\upsilon},$$

$$V_t^e(j,k) = \Lambda_t^e C_t^e(j,k)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{\varsigma_t w_t}\right)^{1-\upsilon},$$

and then make the appropriate substitutions and simplify to obtain an expression for the intertemporal Euler equation:

$$\begin{aligned} &C_t^y(j)\left[\frac{\beta R_t \Omega_{t+1}}{\pi_{t+1}}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)}\right]^{\sigma} \\ &\quad = \omega_{t+1}C_{t+1}^{y}(j) + (1-\omega_{t+1})\frac{\Lambda_{t+1}^{e}}{\Lambda_{t+1}^{y}}C_{t+1}^{e}(j,t+1)\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}, \end{aligned} \tag{A.13}$$

where the adjustment term, Ω_t, is defined as:

$$\Omega_t = \omega_t + (1-\omega_t)\left(\frac{\Lambda_t^e}{\Lambda_t^y}\right)^{1-\rho}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}.$$

Use the conjectured consumption functions, (11) and (22), and substitute these into the consumption Euler equation. Note that we use the fact that an elderly worker born in period j, who just retired at the start of period t, has the following consumption function:

$$C_t^e(j,t) = \xi_t^e\left[\frac{R_{t-1}}{\pi_t}A_{t-1}^{y}(j,t) + H_t^e(j,t) + S_t^e(j,t)\right].$$

After algebraic rearranging we can write the consumption Euler equation as:

$$
\begin{aligned}
&\omega_{t+1}\left(A_t^y(j)+\frac{H_{t+1}^y(j)+S_{t+1}^y(j)}{R_t/\pi_{t+1}}\right)\\
&\quad+(1-\omega_{t+1})\left(\frac{\Lambda_{t+1}^e}{\Lambda_{t+1}^y}\right)\Xi_{t+1}\left(A_t^y(j)+\frac{H_{t+1}^e(j,t+1)+S_{t+1}^e(j,t+1)}{R_t/\pi_{t+1}}\right)\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\\
&\quad=\frac{\xi_t^y}{\xi_{t+1}^y}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^y(j)+H_t^y(j)+S_t^y(j)\right)\left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1}\left[\beta\Omega_{t+1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)}\right]^{\sigma},
\end{aligned}
$$

where $\Xi_t=\xi_t^e/\xi_t^y$. Using the definition of Ω_t we can simplify the above expression as:

$$
\begin{aligned}
&A_t^y(j)+\omega_{t+1}\frac{H_{t+1}^y(j)+S_{t+1}^y(j)}{\Omega_{t+1}R_t/\Pi_{t+1}}\\
&\quad+(1-\omega_{t+1})\left(\frac{\Lambda_{t+1}^e}{\Lambda_{t+1}^y}\right)^{1-\rho}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{H_{t+1}^e(j,t+1)+S_{t+1}^e(j,t+1)}{\Omega_{t+1}R_t/\pi_{t+1}}\\
&\quad=\frac{\xi_t^y}{\xi_{t+1}^y}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^y(j)+H_t^y(j)+S_t^y(j)\right)\beta^{\sigma}\left(\frac{R_t\Omega_{t+1}}{\pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma},
\end{aligned}
\tag{A.14}
$$

if $\Xi_{t+1}=(\Lambda_{t+1}^e/\Lambda_{t+1}^y)^{-\rho}$. From (A.9) we have that $\xi_{t+1}^e=(\Lambda_{t+1}^e)^{-\rho}$.

It remains that we need to verify $\xi_{t+1}^y=(\Lambda_{t+1}^y)^{-\rho}$. Begin by using the budget constraint, (15), and the guessed consumption function of a young worker, (22), to write:

$$
\xi_t^y\left[\frac{R_{t-1}}{\pi_t}A_{t-1}^y(j)+H_t^y(j)+S_t^y(j)\right]+A_t^y(j)+T_t^y(j)=\frac{R_{t-1}}{\pi_t}A_{t-1}^y(j)+w_tL_t^y(j).
$$

Then use the definitions for the present values of a young worker's non-financial assets and social security, (20) and (21), to write the above expression as:

$$
\begin{aligned}
&A_t^y(j)+\omega_{t+1}\frac{H_{t+1}^y(j)+S_{t+1}^y(j)}{\Omega_{t+1}R_t/\Pi_{t+1}}\\
&\quad+(1-\omega_{t+1})\left(\frac{\Lambda_{t+1}^e}{\Lambda_{t+1}^y}\right)^{1-\rho}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{H_{t+1}^e(j,t+1)+S_{t+1}^e(j,t+1)}{\Omega_{t+1}R_t/\pi_{t+1}}\\
&\quad=(1-\xi_t^y)\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^y+H_t^y(j)+S_t^y(j)\right).
\end{aligned}
$$

which is the law of motion of assets for a young worker. Substitute this expression into the intertemporal Euler equation, (A.14), to then get the MPC of young workers:

$$
\frac{1}{\xi_t^y}=1+\beta^{\sigma}\left(\frac{R_t\Omega_{t+1}}{\Pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}\frac{1}{\xi_{t+1}^y}.
$$

Then check the validity of the value function, (17), by writing:

$$\left[\Lambda_t^y C_t^y(j)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{w_t}\right)^{1-\upsilon}\right]^\rho = \left[C_t^y(j)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{w_t}\right)^{1-\upsilon}\right]^\rho$$

$$+\beta\left[\omega_{t+1}\Lambda_{t+1}^y C_{t+1}^y(j)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{w_{t+1}}\right)^{1-\upsilon} + (1-\omega_{t+1})\Lambda_{t+1}^e C_{t+1}^e(j,t+1)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{\varsigma_t w_{t+1}}\right)^{1-\upsilon}\right]^\rho .$$

Combine this expression with the intertemporal Euler equation, (A.13), to yield:

$$(\Lambda_t^y)^\rho = 1 + \beta^\sigma\left(\frac{R_t\Omega_{t+1}}{\pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}(\Lambda_{t+1}^y)^\rho .$$

This expression, as in the case of elderly workers, implies that:

$$\Lambda_t^y = (\xi_t^y)^{\frac{\sigma}{1-\sigma}} .$$

This concludes our verification of Equation (A.14), and we can also write Ω_t as:

$$\Omega_t = \omega_t + (1-\omega_t)\Xi_t^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon} .$$

B Equilibrium conditions

The competitive equilibrium is a sequence of 14 aggregate quantities $\{Y_t, C_t, I_t, K_t, L_t, D_t^I, D_t^K, H_t, A_t, B_t, G_t, E_t, T_t\}$; nine prices $\{w_t, r_t^K, P_t^I, P_t^K, \varphi_t, R_t, \pi_t, R_t^K, Q_t\}$; nine adjustment factors $\{n_t, \gamma_t, \omega_t, \Gamma_t, \Psi_t, \varsigma_t, \Xi_t, \Omega_t, \varrho_t\}$; seven variables for elderly workers $\{\xi_t^e, C_t^e, A_t^e, S_t^e, L_t^e, H_t^e, E_t^e\}$; and seven variables for young workers $\{\xi_t^y, C_t^y, A_t^y, S_t^y, L_t^y, H_t^y, T_t^y\}$. We specify the equilibrium conditions below.

Additionally, as the model features trend technology and population growth, we detrend endogenous variables as follows. The variables Y, C, I, K, P^I, P^K, D^I, D^K, H, A, B, G, E, and T are detrended by XN; L by N; and w by X.

Households. Law of motion for dependency ratio:

$$(1+n_t)\Gamma_t = (1-\omega_t) + \gamma_t \Gamma_{t-1}. \tag{B1}$$

Distribution of wealth:

$$\begin{aligned}&[\Psi_t - (1-\omega_{t+1})]\, a_t = \\ &\omega_{t+1}\left[(1-\xi_t^e)\frac{R_{t-1}}{\pi_t}\frac{\Psi_{t-1}a_{t-1}}{(1+n_t)(1+x_t)} + \varsigma_t w_t l_t^e + e_t^e - \xi_t^e\left(h_t^e + s_t^e\right)\right].\end{aligned} \tag{B2}$$

Ratio of MPCs:

$$\Xi_t = \frac{\xi_t^e}{\xi_t^y}. \tag{B3}$$

Young worker adjustment factor:

$$\Omega_t = \omega_t + (1-\omega_t)\Xi_t^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}. \tag{B4}$$

Elder worker MPC:

$$\frac{1}{\xi_t^e} = 1 + \gamma_{t+1}\beta^{\sigma}\left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1}\left[\frac{\varsigma_t w_t}{(1+x_t)\varsigma_{t+1}w_{t+1}}\right]^{\rho(1-\upsilon)\sigma}\frac{1}{\xi_{t+1}^e}, \tag{B5}$$

Elder worker consumption:

$$c_t^e = \xi_t^e\left[\frac{R_{t-1}}{\pi_t}\frac{a_{t-1}^e}{\gamma_t(1+n_t)(1+x_t)} + h_t^e + s_t^e\right].$$

Elder worker asset proportion:

$$\Psi_t = \frac{a_t^e}{a_t}. \tag{B6}$$

Elder worker pension receipts:

$$s_t^e = e_t^e + \frac{\pi_{t+1}}{R_t}\gamma_{t+1}s_{t+1}^e. \tag{B7}$$

Elder worker capitalised human wealth:

$$h_t^e = \varsigma_t w_t l_t^e + (1 + n_t)(1 + x_t)\frac{\pi_{t+1}}{R_t}\gamma_{t+1}h_{t+1}^e. \tag{B8}$$

Elder worker labour supply:

$$l_t^e = \Gamma_t - \frac{1-\upsilon}{\upsilon}\frac{1}{\varsigma_t w_t}c_t^e. \tag{B9}$$

Young worker MPC:

$$\frac{1}{\xi_t^y} = 1 + \beta^\sigma \left(\frac{R_t\Omega_{t+1}}{\pi_{t+1}}\right)^{\sigma-1}\left[\frac{w_t}{(1+x_{t+1})w_{t+1}}\right]^{\rho(1-\upsilon)\sigma}\frac{1}{\xi_{t+1}^y}. \tag{B10}$$

Young worker consumption:

$$c_t^y = \xi_t^y\left(\frac{R_{t-1}}{\pi_t}\frac{a_{t-1}^y}{(1+n_t)(1+x_t)} + h_t^y + s_t^y\right).$$

Young worker asset ratio:

$$1 - \Psi_t = \frac{a_t^y}{a_t}. \tag{B11}$$

Young worker pension payments:

$$s_t^y = \frac{\omega_{t+1}}{(1+n_{t+1})}\frac{\pi_{t+1}}{R_t\Omega_{t+1}}s_{t+1}^y + \frac{(1-\omega_{t+1})}{(1+n_{t+1})}\Xi_{t+1}^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{\pi_{t+1}}{R_t\Omega_{t+1}}s_{t+1}^e - t_t^y. \tag{B12}$$

Young worker capitalised human wealth:

$$h_t^y = w_t l_t^y + \frac{\omega_{t+1}}{(1+n_{t+1})}\frac{\pi_{t+1}}{R_t\Omega_{t+1}}h_{t+1}^y + \frac{(1-\omega_{t+1})}{(1+n_{t+1})}\Xi_{t+1}^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{\pi_{t+1}}{R_t\Omega_{t+1}}h_{t+1}^e. \tag{B13}$$

Young worker labour supply:

$$l_t^y = 1 - \frac{1-\upsilon}{\upsilon}\frac{1}{w_t}c_t^y. \tag{B14}$$

Firms and production. Aggregate output:

$$y_t = Z_t\left[\frac{k_{t-1}}{(1+x_t)(1+n_t)}\right]^\alpha l_t^{1-\alpha}. \tag{B15}$$

Capital-labour ratio:

$$\frac{(1+x_t)(1+n_t)w_t l_t}{r_t^K k_{t-1}} = \frac{1-\alpha}{\alpha}. \tag{B16}$$

Marginal cost:

$$\varphi_t = \frac{1}{Z_t}\left(\frac{r_t^K}{\alpha}\right)^{\alpha}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}. \tag{B17}$$

NKPC:

$$(\pi_t - 1)\pi_t = \frac{\epsilon - 1}{\phi_I}\left(\mathcal{M}\varphi_t + \tau^s - 1\right) + \frac{\pi_{t+1}}{R_t}\frac{(1+n_t)(1+x_t)y_{t+1}}{y_t}(\pi_{t+1}-1)\pi_{t+1}. \tag{B18}$$

Law of motion for capital:

$$k_t = \frac{(1-\delta)k_{t-1}}{(1+n_t)(1+x_t)} + i_t. \tag{B19}$$

Price of equity:

$$\begin{aligned} Q_t &= 1 + \frac{\kappa_I}{2}\left[\frac{(1+n_t)(1+x_t)i_t}{i_{t-1}} - (1+n_t)(1+x_t)\right]^2 \\ &\quad + \kappa_I\left[\frac{(1+n_t)(1+x_t)i_t}{i_{t-1}} - (1+n_t)(1+x_t)\right]\frac{(1+n_t)(1+x_t)i_t}{i_{t-1}} \\ &\quad - \kappa_I\frac{\pi_{t+1}}{R_t}\left[\frac{(1+n_{t+1})(1+x_{t+1})i_{t+1}}{i_t} - (1+n_{t+1})(1+x_{t+1})\right]\left[\frac{(1+n_{t+1})(1+x_{t+1})i_{t+1}}{i_t}\right]^2. \end{aligned}$$

Profits of intermediate goods producers:

$$d_t^I = \left[\mathcal{M} - \varphi_t\right]y_t - \frac{\phi_I}{2}(\pi_t - 1)^2 y_t.$$

Fiscal and monetary policy. Taylor rule:

$$R_t = \bar{R}^{\phi_R}(R_t^n)^{1-\phi_R}\pi_t^{\phi_\pi}.$$

Government budget constraint:

$$\frac{R_{t-1}}{\pi_t}\frac{b_{t-1}}{(1+n_t)(1+x_t)} + e_t + g_t = b_t + t_t. \tag{B20}$$

Government spending:

$$\frac{g_t}{y_t} = s_t^g. \tag{B21}$$

Government debt issuance:

$$\frac{b_t}{y_t} = s_t^b. \tag{B22}$$

Aggregate pension expenditure:

$$e_t = e_t^e. \tag{B23}$$

Aggregate pension contributions:

$$t_t = t_t^y.$$

Aggregate pension payments to elderly:

$$e_t = \varrho_t \left(w_t l_t^y - t_t \right),$$

with the net replacement rate:

$$\varrho_t = \bar{\varrho}.$$

Equilibrium and aggregation. Resource constraint

$$y_t = c_t + \left[1 + \Phi\left(\frac{i_t}{i_{t-1}} \right) \right] i_t + g_t + \frac{\phi_I}{2}(\pi_t - 1)^2 y_t. \tag{B24}$$

Aggregate consumption:

$$c_t = c_t^y + c_t^e.$$

Aggregate capitalised human wealth:

$$h_t = h_t^y + h_t^e.$$

Aggregate labour supply:

$$l_t = l_t^y + \varsigma_t l_t^e. \tag{B25}$$

Aggregate financial assets:

$$a_t = b_t + k_t + p_t^K + p_t^I.$$

No-arbitrage condition capital goods markets:

$$\frac{R_t}{\pi_{t+1}} = R_{t+1}^K.$$

Gross return on capital:

$$R_t^K = \frac{r_t^K + (1-\delta)Q_t}{Q_{t-1}}.$$

No-arbitrage in intermediate goods stocks:

$$\frac{R_t}{\pi_{t+1}} = \frac{(1 + n_{t+1}) \left(1 + x_{t+1} \right) (p_{t+1}^I + d_{t+1}^I)}{p_t^I}.$$

C Model steady state

In this appendix we analytically solve for the model deterministic steady state. Steady state values of a variable, say X_t, are denoted as simply X.

Absent of trend inflation, we have $\pi = Q = Z = 1$. Through individual Euler equations, $R = \beta^{-1} = R^k$. This yields r^k:

$$r^k = R^k - 1 + \delta.$$

Marginal cost, from (B18), is $\varphi = \mathcal{M}^{-1}$. This allows us to pin down w through (B17):

$$w = \frac{(1-\alpha)\varphi^{\frac{1}{1-\alpha}}}{\left(\frac{r^k}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}}.$$

From the law of motion of capital (B19) we get:

$$\frac{i}{k} = 1 - \frac{1-\delta}{(1+n)(1+x)}.$$

Output to capital ratio can be obtained through the aggregate output condition (B15):

$$\frac{y}{k} = \left(\frac{k}{l}\right)^{\alpha-1} \frac{1}{[(1+n)(1+x)]^{\alpha}}.$$

The capital-labour ratio from (B16) is given by:

$$\frac{k}{l} = \frac{\alpha(1+n)(1+x)}{(1-\alpha)} \frac{w}{r^k}. \tag{C1}$$

Then, use the resource constraint (B24) to get c/k:

$$\begin{aligned} \frac{y}{k} &= \frac{c}{k} + \frac{i}{k} + s^g \frac{y}{k} \\ \Longrightarrow \frac{c}{k} &= (1-s^g)\frac{y}{k} - \frac{i}{k}. \end{aligned} \tag{C2}$$

One can then get a relationship between aggregate labour supply and consumption using (B1), (B25), (B14), and (B9):

$$l = 1 + \varsigma\Gamma - c\frac{1-v}{v}\frac{1}{w},$$

with Γ given by

$$\Gamma = \frac{1-\omega}{1+n-\gamma}$$

One can then get k using (C2) and (C1)

$$k = \frac{1 + \varsigma\Gamma}{\frac{l}{k} + \frac{c}{k}\frac{1-\upsilon}{\upsilon}\frac{1}{w}}.$$

With k at hand, one can get y, l, c, and i given the ratios outlined above.

From (B5), one can get ξ^e:

$$\xi^e = 1 - \gamma\beta^\sigma R^{\sigma-1}(1+x)^{-\rho(1-\upsilon)\sigma}.$$

(B3), (B4), and (B10) constitute a system of 3 equations in 3 unknowns, Ω, Ξ, and ξ^y:

$$\Xi = \frac{\xi^e}{\xi^y},$$

$$\Omega = \omega + (1-\omega)\Xi^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma}\right)^{1-\upsilon},$$

$$\xi^y = 1 - \beta^\sigma (R\Omega)^{\sigma-1}(1+x)^{-\rho(1-\upsilon)\sigma},$$

of which, one variable must be numerically computed as there is no closed-form solution for ξ^y.

Assume that l^y is known. This immediately yields l^e through (B25):

$$l^e = \frac{l - l^y}{\varsigma}.$$

Then get c^e and c^y through labour supply conditions for each agent:

$$c^e = \frac{\upsilon\varsigma}{1-\upsilon}(\Gamma - l^e)w,$$

$$c^y = \frac{\upsilon}{1-\upsilon}(\Gamma - l^y)w.$$

Use (B20), (B21), (B22), and (B23) to pin down e and t. Start with the government budget constraint:

$$t = \left[\frac{R}{(1+n)(1+x)} - 1\right] b + e + g,$$

then use:

$$g = s^g y,$$

$$b = s^b y,$$

$$e = \bar{\varrho}\,(wl^y - t)\,.$$

So we then get:

$$e = \frac{\bar{\varrho}}{1+\bar{\varrho}}\left[wl^y - \left(\frac{R}{(1+n)(1+x)} - 1\right) b - g\right].$$

With $e = e^e$, get s^e from (B7)

$$s^e = \frac{e^e}{1 - \frac{\gamma}{R}}.$$

Since l^e is known, get h^e from (B8):

$$h^e = \frac{\varsigma w l^e}{1 - \frac{\gamma(1+n)(1+x)}{R}}.$$

Then, we can get Ψ from (B2):

$$\Psi = \frac{\omega\left[\varsigma w l^e + e^e - \xi^e(h^e + s^e)\right]}{a\left[1 - (1-\omega) - \frac{\omega(1-\xi^e)R}{(1+n)(1+x)}\right]}.$$

Dividends are:

$$d^I = (\mathcal{M} - \varphi)y,$$

hence, price of shares are:

$$p^I = \frac{d^I}{R-1}.$$

Thence, total assets are given by:

$$a = b + p^I + k.$$

Then we can get a^e from (B6):

$$a^e = \Psi a,$$

a^y from (B11):

$$a^y = (1 - \Psi)a,$$

s^y from (B12):

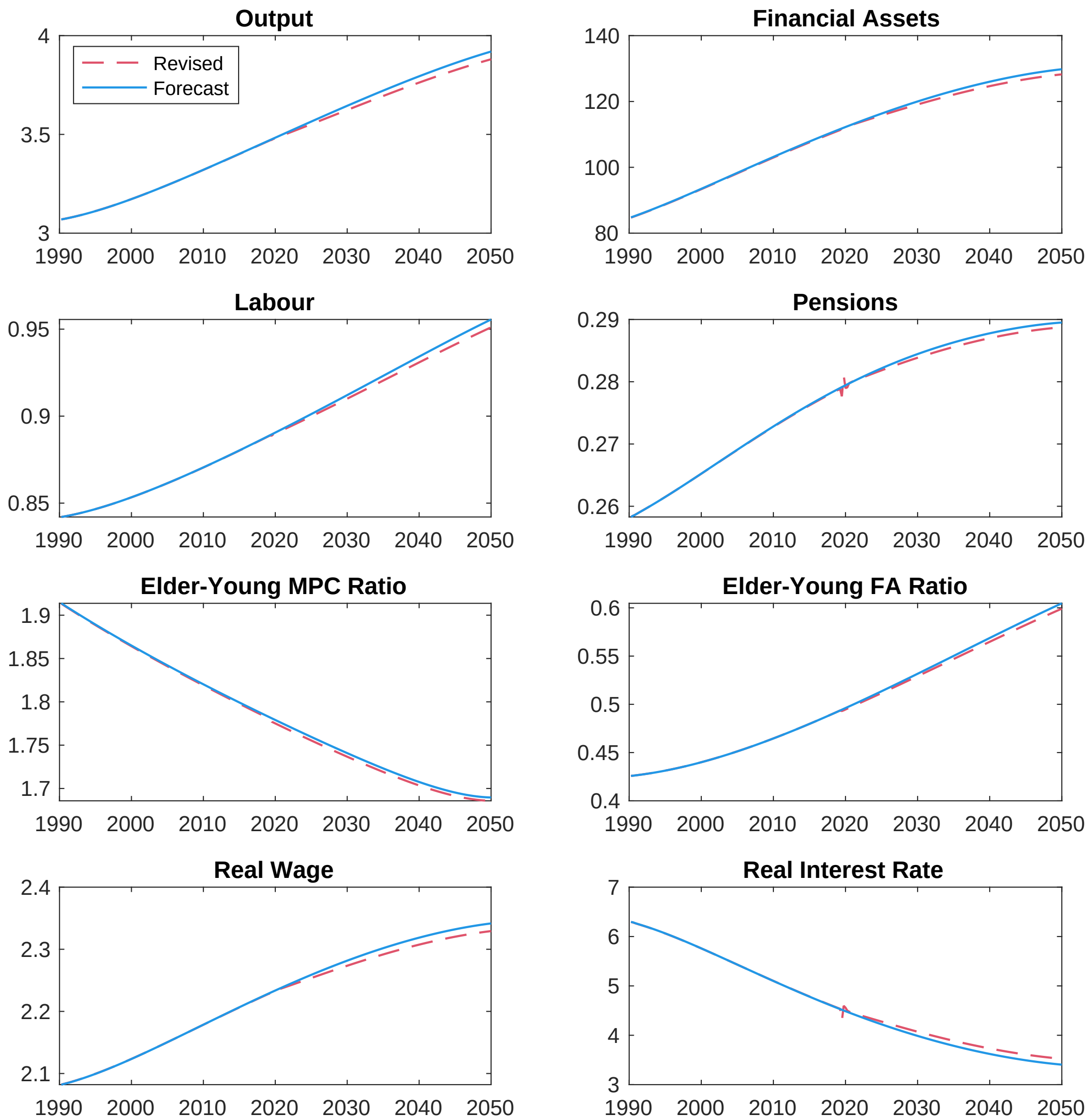
$$s^y = \frac{(1-\omega)\Xi^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma}\right)^{1-\upsilon} s^e - (1+n)R\Omega t^y}{(1+n)R\Omega - \omega},$$

and h^y from (B13):

$$h^y = \frac{(1-\omega)\Xi^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma}\right)^{1-\upsilon} h^e + (1+n)R\Omega w l^y}{(1+n)R\Omega - \omega}.$$

D Additional simulation results

Figure 1: Simulation 1 extension (revision of population forecast)



Note: Figure plots response of variables as n transitions from 0.0011 to -0.0014 and γ transitions from 0.9815 to 9891 based on initial population forecasts (blue). The red dashed line represents the transition path as agents learn that their initial population forecasts were overly pessimistic in 2019-Q1, and that n transitions to -0.0011 at the end of the forecast horizon. The interest rate is expressed in net annualised percentage points.

Simulation 1 extension (positive news about population forecast). In this extension to Simulation 1, we suppose that agents receive positive news about future demographics. For instance, the 2006 release of Japan's IPSS projections for household demographics[1] was overly pessimistic. Figure 1 shows the transition path of key model variables where agents learn in 2019-Q1 that the demographic decline is not as severe as they previously thought. Life expectancy and the retirement age does not change however, in line with Simulation 1 in the main text. The positive news about the population growth rate leads to a decline in financial assets, the elder-young MPC ratio, and the elder-young financial assets ratio as the ratio of elderly-young workers is lower than the baseline Simulation 1 scenario. As such, with less supply of financial assets, there is upward pressure in the real interest rate after a brief period of adjustment in 2019-Q1 in which agents receive the good news shock. But the higher real interest rate and lower capital lead to slightly lower output per capita, as well as a lower real wage since the marginal product is lower too.

[1] The report can be found at: `https://www.ipss.go.jp/site-ad/index_english/population-e.html`.